

## Problems to Introduction to Real Analysis, (Math446)

Due: 09/30/04

Prof.: Thomas Schlumprecht

**Problem 1.** Two sets  $A$  and  $B$  are equipotent if and only if there is an injection  $f : A \rightarrow B$  and a surjection  $g : A \rightarrow B$ .

**Problem 2.** Let  $(X, d)$  be a metric space. For  $F \subset M$  show that:

$A$  is closed  $\iff A$  contains all of its limit points.

**Problem 3.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous and strictly increasing, so that  $a = \lim_{x \rightarrow -\infty} f(x)$  and  $b = \lim_{x \rightarrow +\infty} f(x)$  exist in  $\mathbb{R}$  (for example  $f(x) = \arctan x$ ) for  $x, y \in \mathbb{R}$  define  $d(x, y) = |f(x) - f(y)|$

- $d(\cdot, \cdot)$  is a metric on  $\mathbb{R}$ ,
- The  $d$ -open sets are the same as the usual open sets,
- In  $(\mathbb{R}, d)$  not every Cauchy sequence converges.

**Problem 4.** for  $f, g \in C([0, 1])$ :

$$d_1(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)| \text{ and } d_2(f, g) = \int_0^1 |f(x) - g(x)| dx.$$

- Show that  $d_1$  and  $d_2$  are metrics.
- Show that: If  $(f_n)$  is convergent with respect to  $d_1$  then it is also convergent with respect to  $d_2$ , but that there is a sequence  $(f_n)$  which converges with respect to  $d_2$  but is not convergent with respect to  $d_1$ .

**Problem 5.** Show that  $C([0, 1])$  with the metric  $d_2$  (as in previous problem) is not complete.

**Problem 6.** Let  $(X, d)$  be a metric space. For  $A \subset X$  define

$$A^\circ = \bigcup \{U : U \subset A \text{ open}\} \text{ and } \bar{A} = \bigcup \{F : F \supset A \text{ closed}\}.$$

$A^\circ$  is called *open kernel of A* and  $\bar{A}$  is called *closure of A*.

- $A^\circ$  is open,
- $\bar{A}$  is closed,
- $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$  but not necessarily " $\supset$ ",
- $\bar{A} = A \cup \{x : x \text{ is limit point of } A\}$ .

**\*Problem 7.** Consider the set

$$c_0 = \{(x_i) \subset \mathbb{R} : \lim_{i \rightarrow \infty} x_i = 0\}.$$

(i.e.  $c_0$  is the set of all sequences in  $\mathbb{R}$  which converge to 0).

Note (without writing down proof) that  $c_0$  is a vector space (with  $(x_i) + (y_i) = (x_i + y_i)$  and  $\alpha(x_i) = (\alpha x_i)$ ).

For  $x = (x_i) \in c_0$  define  $\|x\| = \sup_{i \in \mathbb{N}} |x_i| = \max_{i \in \mathbb{N}} |x_i|$ .

- Show that  $\|\cdot\|$  is a norm on  $c_0$
- Show that  $c_0$  is complete