Problems in Real Variables, II (Math608) **Due: 02/17/10**

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Problem 1. Assume that X is a n.l.s. and that Y is a closed proper subspace of X. For $x \in X$ define

$$||x+Y|| = \inf_{y\in Y} ||x+y||.$$

a) Show that map $||\cdot||$ is a norm on X/Y.
b) For any $\varepsilon > 0$ there is an $x \in X$ such that $\|x\| = 1$ and $\|x+Y\| > 1-\varepsilon$.
c) The projection map $P : X \to X/Y$ has norm 1.
d) If X is complete so is X/Y.

Problem 2. Define

$$c_0 = \{(x_i)_{i\in\mathbb{N}} \subset \mathbb{F} : \lim_{i\to\infty} x_i = 0\} \quad (\text{with } \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}).$$

For $x = (x_i)_{i\in\mathbb{N}} \in c_0$ define $\|x\| = \sup_{i\in\mathbb{N}} |x_i|$. Show that $\|\cdot\|$ is norm on c_o and that c_0 is a Banach space.

Problem 3. Problem 2/page 154.
Let X and Y be normed linear spaces Show that $L(X,Y)$ is vector space, that $\|\cdot\|$ with $\|T\| = \sup_{x\in X, \|x\|\le 1} \|T(x)\|$ defines a norm, and show that

$$\|T\| = \sup\Big\{\frac{\|T(x)\|}{\|x\|} : x \neq 0\Big\} = \inf\big\{C \ge 0 : \forall x \in X \quad \|T(x)\| \le C\|x\|\big\}.$$

Problem 4. Problem 9/page 155.
Let $C^k([0,1])$ be the space of all functions on $[0,1]$ which have k-th derivative which is continuous (including halfsided derivatives at the endpoints).

a) for $f \in C([0,1])$ it follows that

$$f \in C^k([0,1]) \iff f \text{ is } k\text{-times cont.diffble and}$$
$$\lim_{h\searrow 0} f^{(j)}(h) \text{ and } \lim_{h\nearrow 1} f^{(j)}(h) \text{ exist for } j \le k.$$

b) For $f \in C^k([0,1])$ put $\|f\| = \sum_{i=0}^k \|f^{(i)}\|_u$. Then with $\|\cdot\|$ the space $C^k([0,1])$ becomes a Banach space.

Problem 5. Problem 6/Page 155.
Let X be a finite dimensional vector space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) and let $e_1, e_2, \dots e_n$ be a basis of X. For $x = \sum_{j=1}^n a_j e_j \in X$, let

$$\|x\|_1 = \sum_{j=1}^n |a_j|.$$

a) $\|\cdot\|_1$ is a norm.

b) The map

$$T : \mathbb{F}^n \to X, \quad (a_1, a_2, \dots a_n) \mapsto \sum_{j=1}^{n} a_j e_j$$

is continuous with respect to Euclidean norm on $\mathbb{F}^n$ and the norm $\| \cdot \|_1$ on X.

c) The *sphere of* $(X, \| \cdot \|_1)$, i.e. the set $\{x \in X : \|x\|_1 = 1\}$ is compact in the topology defined by by $\| \cdot \|$.

d) All norms on X are equivalent.

Problem 6. Problem 7/Page 155.
Let X be a Banach space. We denote the identity on X by I.

a) If $T \in L(X, X)$, and $\|I - T\| < 1$, then T is invertible. In fact the series $\sum_{n=0}^{\infty} (I - T)^n$ (with $(I - T)^0 = I$) converges in $L(X, X)$ to T^{-1}.

b) If $T \in L(X, X)$ is invertible and $\|T - S\| < \|T^{-1}\|^{-1}$, then S is also invertible. Thus, the set of invertible operators in $L(X, X)$ is open in $L(X, X)$.

Problem 7. Let $(X, \mathcal{M})$ be a measurable space and let $M(X)$ be the space of $\mathbb{R}$-valued signedmeasures on $(X, \mathcal{M})$. Then

$$\| \cdot \| : M(X) \to [0, \infty), \quad \mu \mapsto \|\mu\| = |\mu|(X),$$

is a norm on $M(X)$ which turns $M(X)$ into a Banach space.

Problem 8. (Old Qualifier Problem) If $f \in L_2(\mathbb{R})$, $g \in L_3(\mathbb{R})$, and $h \in L_6(\mathbb{R})$, prove that $f \cdot g \cdot h \in L_1(\mathbb{R})$.