Problems in Real Variables, II (Math608), Solutions
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Problem 1. Assume that X is a n.l.s. and that Y is a closed proper subspace of X. For $x \in X$ define

$$||x + Y|| = \inf_{y \in Y} ||x + y||.$$

a) Show that map $|| \cdot ||$ is a norm on X/Y.
b) For any $\varepsilon > 0$ there is an $x \in X$ such that $\|x\| = 1$ and $\|x + Y\| > 1 - \varepsilon$.
c) The projection map $P : X \to X/Y$ has norm 1.
d) If X is complete so is X/Y.

Proof.

a) For $x, z \in X$, and $\alpha \in \mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) it follows that

$$\begin{aligned}\|x + z + Y\| &= \inf_{y \in Y} \|x + z + y\| \\ &= \inf_{y_1, y_2 \in Y} \|x + y_1 + z + y_2\| \\ &\le \inf_{y_1, y_2 \in Y} \|x + y_1\| + \|z + y_2\| \\ &= \inf_{y_1 \in Y} \|x + y_1\| + \inf_{y_2 \in Y} \|z + y_2\| = \|x + Y\| + \|z + Y\|,\end{aligned}$$

and if $\alpha \neq 0$

$$\|\alpha x + Y\| = \inf_{y \in Y} \|\alpha x + y\| = \inf_{y \in Y} \|\alpha x + \alpha y\| = |\alpha| \inf_{y \in Y} \|x + y\| = \alpha \|x + Y\|.$$

If $\alpha = 0$ then it follows also $\|\alpha x + Y\| = 0 = \alpha \|x + Y\|$.
(b) Y is a proper and closed subspace, thus not dense in X. So choose $x_0 \in X$, with $\text{dist}(x_0, Y) = \inf_{y \in Y} \|x_0 - y\| > 0$. Given $\varepsilon > 0$ choose $y_0 \in Y$ so that $\inf_{y \in Y} \|x_0 + y\| > \|x_0 + y_0\|(1 - \varepsilon)$. Finally let $x = (x_0 + y_0)/\|x_0 + y_0\|$. it follows that $\|x\| = 1$ and (by part (a))

$$\begin{aligned}\|x + Y\| &= \|x_0 + y_0\|^{-1} \cdot \|x_0 + y_0 + Y\| \\ &= \|x_0 + y_0\|^{-1} \cdot \inf_{y \in Y} \|x_0 + y_0 + y\| \\ &= \|x_0 + y_0\|^{-1} \cdot \inf_{y \in Y} \|x_0 + y\| > 1 - \varepsilon.\end{aligned}$$

(c) First note that for any $x \in X$

$$\|P(x)\| = \inf_{y \in Y} \|x + y\| \le \|x + 0\| = \|x\|,$$

and thus $\|P\| \le 1$ On the other hand, by part (b), there exists for any $\varepsilon > 0$ an $x \in X$, $\|x\| = 1$, so that $\|x + Y\| \ge \|x\| - \varepsilon = 1 - \varepsilon$, which proves that $\|P\|$ is at least 1.

(d) We are using a result proven in class (Theorem 5.1 on page 158), which says that it is enough to show that an absolutely converging sequence in X/Y converges. So assume that (x_n) is a sequence in X so that

$$\sum_{n\in\mathbb{N}} \|x_n + Y\| < \infty.$$

Choose for $n \in \mathbb{N}$ y_n in Y so that $\|x_n + y_n\| \le \|x_n + Y\| + 2^{-n}$. This yields that $(x_n + y_n)$ is absolute converging in X, and thus that the series $\sum_{n\in\mathbb{N}} x_n + y_n$ converges to some $x \in X$ (X is complete). It follows that

$$\Big\| x - \sum_{j=1}^{n} x_n + Y \Big\| \le \Big\| x - \sum_{j=1}^{n} (x_j + y_j) \Big\| \to_{n\to\infty} 0.$$

Problem 2. Define

$$c_0 = \{(x_i)_{i\in\mathbb{N}} \subset \mathbb{F} : \lim_{i\to\infty} x_i = 0\} \quad \text{(with } \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}).$$

For $x = (x_i)_{i\in\mathbb{N}} \in c_0$ define $\|x\| = \sup_{i\in\mathbb{N}} |x_i|$. Show that $\|\cdot\|$ is norm on c_o and that c_0 is a Banach space.

Proof. Using a result of class (Theorem 5.1 on page 158) we let $(x^{(n)})$ be elements of c_0 for $n = 1, 2, \ldots$, so that $\sum \|x^{(n)}\| < \infty$, and have to show that there is a an $x \in c_0$ so that $\| \sum_{i=1}^{n} x^{(i)} - x\| \to 0$ if $n \to \infty$.

Write for $n \in \mathbb{N}$ $x^{(n)}$ as $(x^{(n)}_m)_{m\in\mathbb{N}} \subset \mathbb{F}$ with $\lim_{m\to\infty} x^{(n)}_m = 0$. For fixed $m \in \mathbb{N}$ we notice that

$$\sum_{n\in\mathbb{N}} |x^{(n)}_m| \le \sum_{n\in\mathbb{N}} \|x^{(n)}\| < \infty.$$

Since $\mathbb{F}$ is complete, we deduce that for each $m \in \mathbb{N}$ there is a number $x_m \in \mathbb{F}$ so that $x_m = \lim_{n\to\infty} \sum_{i=1}^{n} x^{(i)}_m$.

We have to show that $x = (x_m) \in c_0$ and that $\|x - x^{(n)}\|_{c_0} \to 0$, for $n \to \infty$.

Let $\varepsilon > 0$ and we need to find $m \in \mathbb{N}$ so that $|x_k| < \varepsilon$ for all $k \ge m$ (this would prove that $(x_k) \in c_0$). First choose $N \in \mathbb{N}$ so that

$$\sum_{n=N+1}^{\infty} \|x^{(n)} - x^{(n-1)}\| < \varepsilon/2.$$

Then choose $m \in \mathbb{N}$ so that for all $k \ge m$ it follows that $|x^{(N)}_k| \le \varepsilon/2$.

Now note that for all $k \ge m$ we have:

$$\begin{aligned} |x_k| &\le |x_k - x^{(N)}_k| + |x^{(N)}_k| \\ &= \Big| \sum_{n=N+1}^{\infty} x^{(n)}_k - x^{(n-1)}_k \Big| + |x^{(N)}_k| \\ &\le \sum_{n=N+1}^{\infty} \|x^{(n)} - x^{(n-1)}\| + \varepsilon/2 < \varepsilon. \end{aligned}$$

This proves that $x \in c_0$.

In order show that $\lim_{n\to\infty} \|x - x^{(n)}\| = 0$ we let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that $\sum_{j=N}^{\infty} \|x^{(j)} - x^{(j-1)}\| < \varepsilon$. Then it follows for $n \geq N$

$$\|x - \sum_{i=1}^{n} x^{(i)}\| = \sup_{k\in\mathbb{N}} |x_k - \sum_{i=1}^{n} x_k^{(i)}| = \sup_{k\in\mathbb{N}} |\sum_{j=n+1} x_k^{(j)} - x_k^{(j-1)}| \leq \sum_{j=n+1} \|x^{(n+1)} - x^{(n)}\| < \varepsilon.$$

Problem 3. Problem 2/page 154. Let X and Y be normed linear spaces Show that $L(X, Y)$ is vector space, that $\|\cdot\|$ with $\|T\| = \sup_{x\in X, \|x\|\leq 1} \|T(x)\|$ defines a norm, and show that

$$\|T\| = \sup\Big\{\frac{\|T(x)\|}{\|x\|} : x \neq 0\Big\} = \inf\big\{C \geq 0 : \forall x \in X \quad \|T(x)\| \leq C\|x\|\big\}.$$

Proof. Using as addition $S + T : X \to Y$, $x \mapsto T(x) + S(y)$, and as multiplication with scalar $\lambda T : X \to Y$, $x \mapsto \lambda T(x)$, it is clear that $L(X, Y)$ is a vector space.

$\|\cdot\|$ is norm (let $T, S \in L(X, Y)$, $\lambda \in \mathbb{F}$:

$$\begin{aligned} \|T\| = 0 &\iff \sup_{x\in X, \|x\|\leq 1} \|T(x)\| = 0 \\ &\iff \forall x \in X, \|x\| \leq 1, \quad T(x) = 0 \\ &\iff \forall x \in X \quad T(x) = 0 \iff T = 0. \end{aligned}$$

$$\begin{aligned} \|T + S\| &= \sup_{x\in X, \|x\|\leq 1} \|T(x) + S(x)\| \\ &\leq \sup_{x\in X, \|x\|\leq 1} \|T(x)\| + \|S(x)\| \\ &\leq \sup_{x\in X, \|x\|\leq 1} \|T(x)\| + \sup_{x\in X, \|x\|\leq 1} \|S(x)\| = \|T\| + \|S\|. \end{aligned}$$

$\|\lambda T\| = \sup_{x\in X, \|x\|\leq 1} \|\lambda T(x)\| = |\lambda| \sup_{x\in X, \|x\|\leq 1} \|T(x)\| = |\lambda| \|T\|$.

Finally we will show that

$$\|T\| \leq \sup\Big\{\frac{\|T(x)\|}{\|x\|} : x \neq 0\Big\} \leq \inf\big\{C \geq 0 : \|T(x)\| \leq C\|x\|\big\} \leq \|T\|.$$

First "$\leq$": for $x \in X \setminus \{0\}$, $\|T(x)\|/\|x\| = \|T(x/\|x\|)\|$.

Second "$\leq$": Assume that $C \geq 0$ is such that $\forall x \in X \quad \|T(x)\| \leq C\|x\|$. Then $C \geq \|T(x)\|/\|x\|$ for all $x \in X \setminus \{x\}$. Since $C \geq 0$ was arbitrary with $\|T(x)\| \leq C\|x\|$ for all $x \in X$ claim follows.

Third "$\leq$": We have to show that the number $\|T\|$ is element of the set

$$\{C \geq 0 : \forall x \in X \quad \|T(x)\| \leq C\|x\|\}.$$

But this is clear.

Problem 4. Problem 9/page 155.

Let $C^k([0,1])$ be the space of all functions on $[0,1]$ which have k-th derivative which is continuous (including halfsided derivatives at the endpoints).

a) for $f \in C([0,1])$ it follows that

$$f \in C^k([0,1]) \iff f \text{ is } k\text{-times cont.diffble and} \\ \lim_{h \searrow 0} f^{(j)}(h) \text{ and } \lim_{h \nearrow 1} f^{(j)}(h) \text{ exist for } j \le k.$$

b) For $f \in C^k([0,1])$ put $\|f\| = \sum_{i=0}^k \|f^{(i)}\|_u$. Then with $\|\cdot\|$ the space $C^k([0,1])$ becomes a Banach space.

Proof. (a) $\Rightarrow$ is trivial
For $\Leftarrow$ we need to show that

$$\lim_{h \to 0+} \frac{f^{(k-1)}(h) - f^{(k-1)}(0)}{h} = \lim_{h \to 0+} f^{(k)}(h) \text{ and}$$

$$\lim_{h \to 0+} \frac{f^{(k-1)}(1) - f^{(k-1)}(1-h))}{h} = \lim_{h \to 0+} f^{(k)}(1-h).$$

Note that by the MVT we can choose for each $0 < h$ an $r_h \in [0,h]$ so that

$$\frac{f^{(k-1)}(h) - f^{(k-1)}(0)}{h} = f^{(k)}(r_h).$$

And thus

$$\lim_{h \to 0+} \frac{f^{(k-1)}(h) - f^{(k-1)}(0)}{h} = \lim_{h \to 0+} f^{(k)}(r_h) = \lim_{r \to 0} f^{(k)}(r).$$

Similarly the second equation can be show.
(b) It is clear that $C^{(n)}([0,1])$ is a normed linear space. We need to show completeness.

If $(f_n) \in C^{(1)}([0,1])$ is such that

$$\sum_{n \in \mathbb{N}} \|f_n\|_u < \infty \text{ and } \sum_{n \in \mathbb{N}} \|f_n'\|_u < \infty$$

Then it first follows from the completness of $C([0,1])$ that $\sum_{n \in \mathbb{N}} f_n$ converges in $C([0,1])$ to some $f \in C([0,1])$ but also that $\sum_{n \in \mathbb{N}} f_n'$ converges to some $g \in C([0,1])$. We claim that f is differentiable and its derivative is g (what else could it be).

Indeed, let $\varepsilon > 0$ and first choose $N \in \mathbb{N}$ so that $\sum_{n=N}^\infty \|f_n'\|_u < \varepsilon/2$.

Then we note that for $x \in [0,1]$, and $h \neq 0$, such that $x+h \in [0,1]$, it follows that:

$$\begin{aligned}
&\Big|\frac{f(x+h)-f(x)}{h} - g(x)\Big| \\
&\le \Big|\sum_{i=1}^{N} \frac{f_i(x+h)-f_i(x)}{h} - f_i'(x)\Big| + \sum_{i=N+1} \Big|\frac{f_i(x+h)-f_i(x)}{h}\Big| \\
&\qquad + \sum_{i=N+1} |f_i'(x)| \\
&\le \Big|\sum_{i=1}^{N} \frac{f_i(x+h)-f_i(x)}{h} - f_i'(x)\Big| + \sum_{i=N+1} |f_i'(\xi_i)| + \sum_{i=N+1} |f_i'(x)| \\
&\text{[Note: we are applying MVT]} \\
&\le \Big|\sum_{i=1}^{N} \frac{f_i(x+h)-f_i(x)}{h} - f_i'(x)\Big| + 2\sum_{n=N+1}^{\infty} \|f_n'\|_u \\
&\le \Big|\sum_{i=1}^{N} \frac{f_i(x+h)-f_i(x)}{h} - f_i'(x)\Big| + \varepsilon \to \varepsilon, \text{ if } h \to 0,
\end{aligned}$$

which implies our claim.

Now the problem can be easily shown by induction on k.

Problem 5. Problem 6/Page 155.

Let X be a finite dimensional vector space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) and let $e_1, e_2, \dots e_n$ be a basis of X. For $x = \sum_{j=1}^n a_j e_j \in X$, let

$$\|x\|_1 = \sum_{j=1}^n |a_j|.$$

a) $\|\cdot\|_1$ is a norm.

b) The map

$$T : \mathbb{F}^n \to X, \quad (a_1, a_2, \dots a_n) \mapsto \sum_{j=1}^n a_j e_j$$

is continuous with respect to the Euclidean norm on $\mathbb{F}^n$ and the norm $\|\cdot\|_1$ on X.

c) The *sphere of* $(X, \|\cdot\|_1)$, i.e. the set $S_X = \{x \in X : \|x\|_1 = 1\}$ is compact in the topology defined by by $\|\cdot\|$.

d) All norms on X are equivalent.

Proof. (a) For $x = \sum_{j=1}^n a_j e_j, y = \sum_{j=1}^n b_j e_j \in X$ and a scalar α it follows from the definition of bases

$$\|x\|_1 = 0 \iff \sum_{j=1}^n |a_j| = 0 \iff \forall j \in \{1, 2, \dots n\} \quad a_i = 0 \iff x = 0$$

$$\|x + y\|_1 = \sum_{j=1}^n |a_j + b_j| \leq \sum_{j=1}^n |a_j| + |b_j| = \|x\|_1 + \|y\|_1$$

$$\|\alpha x\|_1 = \sum_{j=1}^n |\alpha a_j| = \ |\alpha| \sum_{j=1}^n |a_j| = |\alpha| \cdot \|x\|_1$$

(b) Denote the Euclidean norm by $\| \cdot \|_2$. For $(a_1, a_2, \dots a_n) \in \mathbb{F}^n$

$$\|T(a_1, a_2, \dots)\|_1 = \sum_{j=1}^n |a_j| \leq n \max_{1 \leq i \leq n} |a_j| \leq n \|(a_1, a - 2, \dots a_n)\|_2.$$

(we will show later that actually $\sum_{j=1}^n |a_j| \leq \sqrt{n} \big(\sum_{j=1}^n |a_j|^2 \big)^{1/2}$). Thus T is bounded and $\|T\| \leq n$.

(c) Let $x^{(k)} = \sum_{j=1}^n a_j^{(k)} e_j \in S_X$, for $k \in \mathbb{N}$ note that for each $j \in \{1, 2, \dots n\}$, $(a_j^{(k)})_{k \in \mathbb{N}} \subset \{\xi \in \mathbb{F} : |\xi| = 1\}$. We find therefore an infinite $N \subset \mathbb{N}$ so that

$$a_j = \lim_{k \in \mathbb{N}, k \to \infty} a_j^{(k))}$$

exists for all $j \in \{12 \dots\}$. Let $x = \sum_{j=1}^n a_j e_j$. Then

$$\|x\|_1 = \sum_{j=1}^n |a_j| = \lim_{k \in \mathbb{N}, k \to \infty} \sum_{j=1}^n |a_j^{(k)}| = 1,$$

and

$$\|x - x^{(k)}\|_1 = \sum_{j=1}^n |a_j - a_j^{(k)}| = 1 \to_{k \in \mathbb{N}, k \to \infty} 0.$$

Thus, every sequence in S_X has a subsequence which converges to an element of S_X. Thus S_X is compact.

(d) Let $\| \cdot \|$ be any norm on X. Put $C = \max_{j=1,2 \dots n} \|e_j\|$. Then (property of norm) $0 < C < \infty$. Let T be the identity on X, but think of it as a linear map from $(X, \| \cdot \|_1)$ to $(X, \| \cdot \|)$.

Since for $x = \sum_{j=1}^n a_j e_j \in X$

$$\|x\| = \|T(x)\| = \Big\| \sum_{j=1}^n a_j e_j \Big\| \leq \sum_{j=1}^n |a_j| \|e_j\| \leq C \|x\|_1$$

T is a bounded linear operator with $\|T\| \leq C$. This implies that the the image of $S_X = \{x \in X : \|x\|_1\}$ is compact in $(X, \| \cdot \|)$. And since $0 \notin S_X$ and since $\| \cdot \|$ is a $\| \cdot \|$-continuous function on X (simply meaning that if

x_n converges to x in $(X, \|\cdot\|)$ then $\|x_n\|$ converges to $\|x\|$) it follows that $c := \min\{\|x\| : x \in S_X\}$ exists and $c > 0$. We deduce that for $x \in X$

$$C\|x\|_1 \ge \|x\| = \|x\|_1 \Big\| \frac{x}{\|x\|_1} \Big\| \ge c\|x\|_1,$$

which proves our claime.

Problem 6. Problem 7/Page 155.
Let X be a Banach space. We denote the identity on X by I.

a) If $T \in L(X, X)$, and $\|I - T\| < 1$, then T is invertible. In fact the series $\sum_{n=0}^{\infty}(I-T)^n$ (with $(I-T)^0 = I$) converges in $L(X, X)$ to T^{-1}.
b) If $T \in L(X, X)$ is invertible and $\|T - S\| < \|T^{-1}\|^{-1}$, then S is also invertible. Thus, the set of invertible operators in $L(X, X)$ is open in $L(X, X)$.

Proof. (a) Let $\rho := \|I - T\| \in [0, 1)$. By induction we can proof that $\|(I-T)^n\| \le \rho^n$. Indeed, for $n = 1$, this is the assumption, if the claim is true for n, it follows for $x \in X$, that

$$\|(I-T)^{n+1}(x)\| \le \|(I-T)^n\| \cdot \|(I-T)(x)\| \le \rho^n \rho \|x\| = \rho^{n+1}\|x\|.$$

Thus the series $\sum_{n=0}^{\infty}(I-T)^n$ is absolutely convergent, and, thus, since $L(X, X)$ is complete, convergent to some $S \in L(X, X)$. We claim that S is the inverse of T, and need to show that $T \circ S = S \circ T = I$. For $x \in X$, we observe that

$$\begin{aligned} S(T(X)) &= \sum_{n=0}^{\infty}(I-T)^n(T(x)) \\ &= \sum_{n=0}^{\infty}(I-T)^n((T-I)(x)) + \sum_{n=0}^{\infty}(I-T)^n \\ &= \sum_{n=0}^{\infty}(I-T)^n(x) - \sum_{n=1}^{\infty}(I-T)^n(x) = x \end{aligned}$$

and

$$\begin{aligned} T(S(X)) &= T\Big(\sum_{n=0}^{\infty}(I-T)^n(x)\Big) \\ &= \sum_{n=0}^{\infty}(T-I)(I-T)^n(x) + \sum_{n=0}^{\infty}(I-T)^n \\ &= \sum_{n=0}^{\infty}(I-T)^n(x) - \sum_{n=1}^{\infty}(I-T)^n(x) = x \end{aligned}$$

which proves our claim.
(b) We first note that,

$$\|I - T^{-1} \circ S\| = \|T^{-1} \circ T - T^{-1} \circ S\| \le \|T^{-1}\| \cdot \|T - S\| < 1.$$

By part (a), $T^{-1}S$ is invertible with continuous inverse, and thus also $T(T^{-1}S) = S$.

Problem 7. Let $(X, \mathcal{M})$ be a measurable space and let $M(X)$ be the space of real-valued measures on $(X, \mathcal{M})$. Then

$$\|\cdot\| : M(X) \to [0, \infty), \quad \mu \mapsto \|\mu\| = |\mu|(X),$$

is a norm on $M(X)$ which turns $M(X)$ into a Banach space.
Proof. For $\mu \in M(X)$, we put

$$\|\mu\| = |\mu|(X) = \mu^+(X) + \mu^-(X) = \mu(X^+) - \mu(X^-),$$

where $(X^+, X-)$ is the by the Hahn Decomposition Theorem existing partition of X so that $\mu(X^+ \cap (\cdot))$ and $-\mu(X^- \cap (\cdot))$ are positive measures. If μ and $\tilde{\mu}$ are two $\mathbb{R}$-valued signed measure. Choose by Hahn Decomposition Theorem Partitions (X^+, X^-), $(\tilde{X}^+, \tilde{X}^-)$, $(\overline{X}^+, \overline{X}^-)$ of X so that

$$\begin{aligned}
&\mu^+(\cdot) = \mu(X^+ \cap (\cdot)), \quad \mu^-(\cdot) = -\mu(X^- \cap (\cdot)),\\
&\tilde{\mu}^+(\cdot) = \tilde{\mu}(\tilde{X}^+ \cap (\cdot)), \quad \tilde{\mu}^-(\cdot) = -\tilde{\mu}(\tilde{X}^- \cap (\cdot)).\\
&(\mu + \tilde{\mu})^+(\cdot) = (\mu + \tilde{\mu})^+(\overline{X}^+ \cap (\cdot)), \quad (\mu + \tilde{\mu})^-(\cdot) = -(\mu + \tilde{\mu})^-(\overline{X}^- \cap (\cdot)),
\end{aligned}$$

Then

$$\begin{aligned}
||\mu + \tilde{\mu}|| &= |\mu + \tilde{\mu}|(X)\\
&= (\mu + \tilde{\mu})(\overline{X}^+) - (\mu + \tilde{\mu})(\overline{X}^-)\\
&= \mu(\overline{X}^+) + \tilde{\mu}(\overline{X}^+) - \mu(\overline{X}^-) - \tilde{\mu}(\overline{X}^-)\\
&= \mu^+(\overline{X}^+) - \mu^-(\overline{X}^+) + \tilde{\mu}^+(\overline{X}^+) - \tilde{\mu}^-(\overline{X}^+)\\
&\qquad - \mu^+(\overline{X}^-) + \mu^-(\overline{X}^+) - \tilde{\mu}^+(\overline{X}^-) + \tilde{\mu}^-(\overline{X}^+)\\
&\leq \mu^+(X) + \tilde{\mu}^+(X) + \mu^-(X) + \tilde{\mu}^-(X) = \|\mu\| + \|\tilde{\mu}\|.
\end{aligned}$$

Secondly for $\alpha \in \mathbb{R}$

$$\begin{aligned}
\|\alpha\mu\| &= (\alpha\mu)^+(X) + (\alpha\mu)^-(X)\\
&= \begin{cases} \alpha\mu(X^+) + \alpha\mu^-(X) & \text{if } \alpha \geq 0 \\ -\alpha\mu^-(X) - \alpha^+(X) & \text{if } \alpha < 0 \end{cases} = |\alpha| \cdot \|\mu\|
\end{aligned}$$

Finally

$$\|\mu\| = 0 \iff \mu^+(X) = 0 \text{ and } \mu^-(X) = 0 \iff \mu = 0.$$

This proves that $\|\cdot\|$ is a norm.

In order to show completeness we assume that $(\mu_n) \subset M(X)$ and that $\sum \mu_n$ is absolutely converging. This implies that $\sum_{n=1}^\infty \mu^+(X) < \infty$ and $\sum_{n=1}^\infty \mu^-(X) < \infty$. For $A \in \mathcal{M}$ put $\nu_1(A) = \sum_{n\in\mathbb{N}} \mu_n^+(A)$. we claim that ν_1 is a (positive) measure and that $\|\nu_1 - \sum_{j=1}^n \mu_j^+\| \to 0$, for $n \to \infty$.

If (A_i) are pairwise disjoint sets in $\mathcal{M}$, $A = \bigcup_{i=1}^{\infty} A_i$ and $\varepsilon > 0$ arbitrary, we choose $n \in \mathbb{N}$ so that $\sum_{j=n+1}^{\infty} \mu_n^+(X) < \varepsilon$, then

$$\begin{aligned}|\nu_1(A) - \sum_{i=1}^{\infty} \nu_1(A_i)| &= |\sum_{j=1}^{\infty} \mu_j^+(A) - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu_j^+(A_i)| \\ &\le |\sum_{j=1}^{n} \mu_j^+(A) - \sum_{i=1}^{\infty}\sum_{j=1}^{n} \mu(A_i)| + \sum_{j=n+1}^{\infty} \mu_j^+(A) + \sum_{j=n+1}^{\infty}\sum_{i=1}^{\infty} \mu_j^+(A_i) \\ &\le 0 + \varepsilon/2 + \sum_{j=n+1}^{\infty} \mu_j^+(A) \le \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\nu_1(A) = \sum_{i=1}^{\infty} \nu_1(A_i)$, which implies that ν_1 is a positive measure with $\nu_1(X) = \sum_{j=1}^{\infty} \mu_j^+(X) \le \sum_{j=1}^{\infty} \|\mu_n\| < \infty$.

Moreover $\nu_1 - \sum_{j=1}^{n} \mu_j^+$ is a positive measure for all $n \in \mathbb{N}$ and

$$\|\nu_1 - \sum_{j=1}^{n} \mu_j^+\| = \sum_{j=n+1}^{\infty} \mu_n^+(X) \to_{n\to\infty} 0.$$

In a similar way one can show that there is a positive finite measure ν_2 so that $\|\nu_2 - \sum_{j=1}^{n} \mu_j^-\| \to 0$ if $n \to 0$.

Letting $\nu(\cdot) = \nu_1(\cdot) - \nu_2(\cdot)$, it follows that

$$\nu = \nu_1 - \nu_2 = \lim_{n\to\infty} \sum_{j=1}^{n} \mu_j^+ - \mu_j^- = \lim_{n\to\infty} \sum_{j=1}^{n} \mu_j,$$

which finishes the proof of our claim.

Problem 8. (Old Qualifier Problem) If $f \in L_2(\mathbb{R})$, $g \in L_3(\mathbb{R})$, and $h \in L_6(\mathbb{R})$, prove that $f \cdot g \cdot h \in L_1(\mathbb{R})$.
Proof. Let $f \in L_2(\mathbb{R})$, $g \in L_3(\mathbb{R})$, and $h \in L_6(\mathbb{R})$, then

$$\begin{aligned}&\int |f(x)| \cdot |g(x)| \cdot |h(x)|\, dx \\ &\quad \le \left(\int |f(x)|^{\frac{5}{6}} \cdot |g(x)|^{\frac{6}{5}}\, dx\right)^{\frac{5}{6}} \left(\int |h(x)|^6\, dx\right)^{\frac{1}{6}}\end{aligned}$$

(Apply Theorem of Hölder to the functions fg and h and the numbers $p = 6/5$ and $q = 6$)

$$\le \left(\left(\int |f(x)|^{\frac{6}{5}\frac{10}{6}}\, dx\right)^{\frac{6}{10}} \left(\int |g(x)|^{\frac{6}{5}\frac{10}{4}}\, dx\right)^{\frac{4}{10}}\right)^{\frac{5}{6}} \left(\int |h(x)|^6\, dx\right)^{\frac{1}{6}}$$

(Apply Theorem of Hölder to the functions $|f|^{6/5}$ and $|g|^{6/5}$ and the numbers $p = 10/6$ and $q = 10/4$)

$$
\begin{aligned}
&= \left(\int |f(x)|^2 \, dx \right)^{\frac{1}{2}} \left(\int |g(x)|^3 \, dx \right)^{\frac{1}{3}} \left(\int |h(x)|^6 \, dx \right)^{\frac{1}{6}} \\
&= \|f\|_2 \cdot \|g\|_3 \cdot \|h\|_6 .
\end{aligned}
$$