HOMEWORK #4

SECTION 3.1 → #11:

Let $x \in [0,1]$. Then $0 \le x \le 1$. Since f is increasing, $f(0) \le f(x) \le f(1)$, and since $f(0)=1$ and $f(1)=6$, we get $1 \le f(x) \le 6$. So $f(x) \in [1,6]$. This proves that $f(X) \subseteq [1,6]$.

Now suppose $y \in [1,6]$. Then $f(0) \le y \le f(1)$ so by the Intermediate Value Theorem, there exists $x \in [0,1]$ such that $f(x)=y$. Thus $y \in f(X)$, proving that $[1,6] \subseteq f(X)$. Therefore $f(X) = [1,6]$.

SECTION 3.1 → #17:

(a) $f^{-1}(B) = A$

(b) $f^{-1}(\{a,c,d,e\}) = \{1,2,3,4,5\}$.

(c) $f^{-1}(\{a,b,c,d\}) = \{1,2,4,5\}$.

(d) $f^{-1}(\{b\}) = \emptyset$

SECTION 3.1 → #21:

(a) Let $a \in X$. Then $f(a) \in f(X)$. Hence $a \in f^{-1}(f(X))$. Therefore $X \subseteq f^{-1}(f(X))$.

(b) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x)=x^2$. Let $X=\{1\}$. Then $f(X)=\{1\}$ and $f^{-1}(f(X)) = \{-1,1\} \ne X$.

SECTION 3.2 → #6:

(a) Let $b \in B - f(X)$. Then $b \in B$ but $b \notin f(X)$. Since f is surjective, there exists $a \in A$ s.t. $b = f(a)$. Since $b \notin f(X)$, $a \notin X$. Hence $a \in A - X$ and therefore $b \in f(A-X)$. This proves that $B - f(X) \subseteq f(A-X)$.

(b) Let $A = \mathbb{R}$, $B = \{x \in \mathbb{R} \mid x \ge 0\}$ and let $f: A \to B$ be defined by $f(x) = x^2$. Let $X = [0,1]$. Then $f(X) = [0,1]$ and $B - f(X) = (1,\infty)$. But $A - X = (-\infty,0) \cup (1,\infty)$ so $f(A-X) = (0,\infty)$.

SECTION 3.2 → #17:

In exercise 15 of section 3.1 it was proved that $f(X \cap Y) \subseteq f(X) \cap f(Y)$. So it sufficies to prove the opposite inclusion. Let $b \in f(X) \cap f(Y)$. Then $b \in f(X)$ and $b \in f(Y)$. So we can write $b = f(x) = f(y)$ where $x \in X$ and $y \in Y$. Since f is injective, $x = y$ implying that $x \in X \cap Y$. Thus $b \in f(X \cap Y)$. We have proved that $f(X) \cap f(Y) \subseteq f(X \cap Y)$. Therefore $f(X) \cap f(Y) = f(X \cap Y)$.