Problems to Introduction to Real Analysis, (Math446)

Due: 10/14/04

Problem 1. On a test a student was asked to write down the property of a function $d : X \times X \to \mathbb{R}$, which turns d into a metric. She wrote down the following conditions ($x, y, z \in X$ arbitrary):

(i) $d(x, y) = 0 \iff x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

The professor deduced 2 points. Why?

After class she goes to the professor and asserts that these three properties imply that d is a metric. Is she right?

Problem 2. Let $a > 0$, and define $f(x) = x^2 - a$, for $x \in \mathbb{R}$. The Newton method provides a method to solve the equation $f(x) = 0$ (i.e. $\sqrt{a}$).

Let $x_0 > 0$, such that $x_0^2 > a$, be arbitrary and define recursively

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Show that (x_n) is monotone decreasing, that $x = \lim_{n\to\infty} x_n$ exists and that $x^2 = a$.

Problem 3. Regard on $\mathbb{Q}$ as metric space with the usual metric ($d(p, q) = |p - q|$).

Show that the set $\{p : 2 < p^2 < 3\}$ is open as well as closed in $\mathbb{Q}$ but not compact. (You can use without proof that $\sqrt{2}, \sqrt{3} \notin \mathbb{Q}$.

Problem 4.

a) Show that the set of all finite subsets $\mathbb{N}$ is countable.
b) Show that c_0 (as defined on previous homework) is spearable.

Problem 5. Define

$$\ell_\infty = \{(x_i) \subset \mathbb{R} : (x_i) \text{ is bounded}\}.$$

Without writting down proof note that ℓ_∞ is a vectorspace. For $x = (x_i) \in \ell_\infty$ define

$$\|x\| = \sup_{i\in\mathbb{N}} |x_i|.$$

a) Show that $\|\cdot\|$ is a norm on ℓ_∞.
b) Show that ℓ_∞ is complete.
c) Show that ℓ_∞ is not separable.
[Hint: $P(\mathbb{N})$ is uncountable by theorem in class]

Problem 6. Assume that (X, d) is a metric space. Denote the open sets in X by $\mathcal{T}_d$ and the closed sets by $\mathcal{F}_d$. Show that

a) $\mathcal{T}_d$ is closed under taking arbitrary unions, and finite intersections,

b) $\mathcal{F}_d$ is closed under arbitrary intersections, and finite unions.

Problem 7. Let (X, d) be a metric space. Show that for $x \in X$ and $\varepsilon > 0$ the set

$$\{y \in X : d(x, y) \leq \varepsilon\}$$

is closed. Give an example for which

$$U_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\} \neq \{y \in X : d(x, y) \leq \varepsilon\}.$$

Hint: take for example $X = (\infty, 1] \cup \{2\}$.