**Problems in Real Variables, II (Math608)** **Due: 02/24/10**
Prof.: Thomas Schlumprecht

**Problem 1.** (Old Qualifying Exam) Assume we consider on $\mathbb{R}$ the Lebesgues measure.

a) Prove for $1 \le p < \infty$ that the continuous functions on $\mathbb{R}$ with compact support are dense in $L_p(\mathbb{R})$.
b) For $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L_p(\mathbb{R})$, and $g \in L_q(\mathbb{R})$ define

$$f * g(x) = \int f(z) \cdot g(x-z)dz, \ x \in \mathbb{R}.$$

The map $f * g$ is called the *convolution of* $f$ *and* $g$. Show that the function $x \mapsto f*g(x)$ is continuous on $\mathbb{R}$ and that $\sup_{x\in\mathbb{R}} |f*g(x)| \le ||f||_p \cdot ||g||_q$.

**Problem 2.** (The case $p = \infty$) Let $(X, \mathcal{M}, \mu)$ be a measure space. For measurable $f : X \to \mathbb{R}$ (or $\mathbb{C}$ instead of $\mathbb{R}$), we define

$$\|f\|_\infty = \inf\big\{r \ge 0 : \mu(\{x \in X : |f(x)| > a\} = 0\big\},$$

and let

$$L_\infty(X, \mathcal{M}, \mu) = L_\infty(\mu) = \{f : X \to \mathbb{R} : \text{mble.} \|f\|_\infty < \infty\}$$

a) (Theorem of Hölder for $p = \infty$ and $q = 1$) For $f \in L_\infty(\mu)$ and $g \in L_1(\mu)$, we have $f \cdot g \in L_1(\mu)$, and $\|fg\|_1 \le \|f\|_\infty \cdot \|g\|_1$. Characterize when $\|fg\|_1 = \|f\|_\infty \cdot \|g\|_1$.
b) $\|\cdot\|_\infty$ is norm on $L_\infty$,
c) For $(f_n) \subset L_\infty(\mu)$, and $f \in L_\infty(\mu)$, we have that $f_n \to f$ in $\|\cdot\|_\infty$ if and only if there is a $\mu$-null set $E$ so that $f_n$ converges uniformly to $f$ outside of $E$.
d) $L_\infty(\mu)$ is a Banach space.
e) Simple functions are dense in $L_\infty(\mu)$.
f) Continuous bounded functions on $\mathbb{R}$ are not dense in $L_\infty(\mathbb{R})$

**Problem 3.** Let $1 < p < \infty$. Find a function $f : [0,1] \to \mathbb{R}$ which is in $\bigcup_{r<p} L_r[0,1]$, but not in $L_p[0,1]$, and find a function $g \to \mathbb{R}$ which is in $L_p[0,1]$ but not in $\bigcap_{r>p} L_r[0,1]$. **Hint:** $\frac{1}{x} \notin L_1[0,1]$.

**Problem 4.** Let $1 \le p < \infty$. For sequence $(f_n) \subset L_p(\mu)$ ( $(X, \mathcal{M}, \mu)$ measure space) and $f \in L_p(\mu)$ it follows that

$$f_n \to_{n\to\infty} f \text{ in } L_p \iff f_n \to f \text{ in measure and } \|f_n\|_p \to_{n\to\infty} \|f\|_p.$$

**Hint:** First show that $\lim_{n\to\infty} \|f + f_n\|_p = 2\|f\|_p$ and then show that for any $\varepsilon > 0$ there is a $C > 0$ so that $\|f_n\chi_{\{|f_n|>C|f|\}}\|p < \varepsilon$.

**Problem 5.** A linear functional on a normed linear space is bounded if and only if $f^{-1}(\{0\})$ is closed.

**Problem 6.** Suppose $X$ and $Y$ are normed linear spaces and $T \in L(X, Y)$. Define $T^t : Y^* \to X^*$ by $y^* \mapsto y^* \circ T$ (which must be in $X^*$ since $y^*$ and $T$ are both bounded and linear).

a) $T \in L(Y^*, X^*)$ and $\|T^t\| = \|T\|$.
b) Let $I_X : X \to X^{**}$ and $I_X : Y \to Y^{**}$ be the canonical embeddings, then $T^{tt} \circ I_X = I_Y \circ T$.
c) $T^t$ injective $\iff \overline{T(X)} = Y$.
d) If the range of $T^t$ is dense in $X^*$, then $T$ is injective. If $X$ is reflexive the converse holds.

**Problem 7.** Problem 18/page 159 Let $X$ be a normed linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

a) If $Y$ is a closed subspace and $x \in X \setminus Y$ then $x\mathbb{F} + Y$ is closed
b) Every finite dimensional subspace of $X$ is closed.

**Problem 8.** Problem 19/page 160 Let $X$ be an infinite-dimensional normed vector space.

a) There is a sequence $(x_n) \subset X$, with $\|x_n\| = 1$, for all $j \in \mathbb{N}$, and $\|x_j - x_i\| \geq \frac{1}{2}$, if $i \neq j$.
b) The unit ball $B_X = \{x \in X, \|x\| \leq 1\}$ is not compact.

**Remark.** By a much deeper Theorem due to Dor and Odell the following is true: For very infinite dimensional Banach space $X$ there is an $\varepsilon = \varepsilon_X > 0$, so that there is a sequence $(x_j) \in S_X = \{x \in X : \|x\| = 1\}$ so that $\|x_i - x_j\| \geq 1 + \varepsilon$, if $i \neq j$.

See if you can proof the following weaker statement: There is a sequence $(x_j) \subset S_X$ so that $\|x_i - x_j\| > 1$, if $i \neq j$ (this is not a required homework).