Problems in Real Variables, II (Math608), Solutions
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Problem 1. (Old Qualifying Exam) Assume we consider on $\mathbb{R}$ the Lebesgues measure.

a) Prove for $1 \le p < \infty$ that the continuous functions on $\mathbb{R}$ with compact support are dense in $L_p(\mathbb{R})$.
b) For $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L_p(\mathbb{R})$, and $g \in L_q(\mathbb{R})$ define

$$f * g(x) = \int f(z) \cdot g(x - z) dz, \ x \in \mathbb{R}.$$

The map $f * g$ is called the *convolution of f and g*. Show that the function $x \mapsto f*g(x)$ is continuous on $\mathbb{R}$ and that $\sup_{x\in\mathbb{R}} |f*g(x)| \le ||f||_p \cdot ||g||_q$.

Proof. First note that every $f \in L_p$ is the L_p-limit of elements in L_p with compact support. Indeed choose $f_n = \chi_{[-n,n]}f$, then f_n converges pointwise to f, secondly $|f - f_n|^p \le 2^p|f|$, thus by the Dominated Convergence Theorem $\|f - f_n\|_p \to 0$ for $n \to \infty$. Secondly, note that the bounded measurable functions with compact support are dense in L_p. This follows from the fact that the simple functions are dense (as shown in class).

Now let $n \in \mathbb{N}$ and let ($\mathcal{B}_{[-n,n]}$ denote the Borel sets in $[-n, n]$, and $C_c(\mathbb{R})$ are the continous functions on $\mathbb{R}$ with compact support)

$$\mathcal{D} = \{A \in \mathcal{B}_{[-n,n]} : \forall \varepsilon > 0 \exists f \in C_c(\mathbb{R}) : \|\chi_A - f\|_p < \varepsilon\}.$$

It is easy to see that $\mathcal{D}$ is a Dynkin system which contains the subintervals of $[-n, n]$. Since step functions with bounded support are dense in $L_p(\mathbb{R})$, we deduce our claim.

(b) First note that if $g \in L_q$ then for each $x \in \mathbb{R}$ that the map $g(\cdot - x) : \mathbb{R} \ni z \mapsto g(z - x)$ is in L_q and $\|g(\cdot - x)\|_q = \|g\|_q$ (translation invariance of Lebesgues measure) Therefore, it follows from the Theorem of Hölder that for all $x \in \mathbb{R}$

$$|f * g(x)| \le \|f\|_p \cdot \|g(\cdot - x)\|_q = \|f\|_p \cdot \|g\|_q,$$

which means that $f * g$ is bounded.

If g is continuous with compact support it follows again from the Theorem of Hölder for $x, y \in \mathbb{R}$, that

$$|f * g(x) - f * g(y)| \le \|f\|_p \cdot \|g(\cdot - x) - g(\cdot - y)\|_q.$$

Since continuous functions on compact spaces are uniformely continuous it follows that $f * g$ is continuous if g is continuous with compact support.

Assume now that $g \in L_q$ is general, let $x \in \mathbb{R}$ and $\varepsilon > 0$. First choose by (a) a continuous $\tilde{g}$ with compact support so that $\|g - \tilde{g}\|_q < \varepsilon/3(1 + \|f\|_p)$ then choose $\delta > 0$ so that for all $z \in \mathbb{R}$ with $|x - z| < \delta$ it follows that

$|f * \tilde{g}(x) - f * \tilde{g}(z)| < \varepsilon/3$. For all $z \in \mathbb{R}$ with $|x - z| < \delta$ it then follows that

$$\begin{aligned}|f * g(x) - f * g(z)| &\leq |f * g(x) - f * \tilde{g}(x)| + |f * \tilde{g}(x) - f * \tilde{g}(z)| + |f * \tilde{g}(z) - f * g(z)| \\ &\leq 2\|f\|_p \|g - \tilde{g}\|_q + \varepsilon/3 < \varepsilon.\end{aligned}$$

Problem 2. (The case $p = \infty$) Let $(X, \mathcal{M}, \mu)$ be a measure space. For measurable $f : X \to \mathbb{R}$ (or $\mathbb{C}$ instead of $\mathbb{R}$), we define

$$\|f\|_\infty = \inf \big\{ r \geq 0 : \mu(\{x \in X : |f(x)| > r\} = 0 \big\},$$

and let

$$L_\infty(X, \mathcal{M}, \mu) = L_\infty(\mu) = \{ f : X \to \mathbb{R} : \text{mble.} \|f\|_\infty < \infty \}$$

a) (Theorem of Hölder for $p = \infty$ and $q = 1$) For $f \in L_\infty(\mu)$ and $g \in L_1(\mu)$, we have $f \cdot g \in L_1(\mu)$, and $\|fg\|_1 \leq \|f\|_\infty \cdot \|g\|_1$. Characterize when $\|fg\|_1 = \|f\|_\infty \cdot \|g\|_1$.
b) $\|\cdot\|_\infty$ is norm on L_∞,
c) For $(f_n) \subset L_\infty(\mu)$, and $f \in L_\infty(\mu)$, we have that $f_n \to f$ in $\|\cdot\|_\infty$ if and only if there is a μ-null set E so that f_n converges uniformly to f outside of E.
d) $L_\infty(\mu)$ is a Banach space.
e) Simple functions are dense in $L_\infty(\mu)$.
f) Continuous bounded functions on $\mathbb{R}$ are not dense in $L_\infty(\mathbb{R})$

Proof. (a) Let $f \in L_\infty(\mu)$ and $g \in L_1(\mu)$. For an arbitrary $\varepsilon > 0$, it follows from the definition of $\|f\|_\infty$ that the set

$$A = \{x \in X : |f(x)| \geq \varepsilon + \|f\|_\infty\}$$

is a μ-Nullset. Thus it follows that

$$\int |g(x)f(x)| \leq \int |g(x)| \cdot (\|f\|_\infty + \varepsilon) \, d\mu = \|g\|_1 (\|f\|_\infty + \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary we deduce the claim.

Assume that $||gf||_1 = \|g_1\|_1 \|f\|_\infty$ and thus

$$\int (\|f\|_\infty - f(x)) g(x) \, d\mu = 0.$$

Since $f \leq \|f\|_\infty$ μ a.e. it follows that $f = \|f\|_\infty$, μ-a.e. Conversely, if $f = \|f\|_\infty$, μ-a.e. it follows clearly that $\|fg\|_1 = \|g\|_1$.
(b) For $f, g \in L_\infty(\mu)$ and a scalar α we observe:

$$\|f\|_\infty = 0 \iff \forall r > 0, \quad |f| < r \ \mu\text{-a.e.} \iff f = 0 \ \mu\text{-a.e.},$$

$$\begin{aligned}\|\alpha f\|_\infty &= \inf\big\{r>0:\mu(\{x\in X:|\alpha f(x)|>r\}=0\big\}\\ &=\begin{cases}0 & \text{if } \alpha=0\\ \inf\big\{r>0:\mu(\{x\in X:|f(x)|>r/\alpha\}=0\big\} & \text{if } \alpha\neq 0\end{cases}\\ &=\begin{cases}0 & \text{if } \alpha=0\\ \inf\big\{|\alpha|\rho:\rho>0:\mu(\{x\in X:|f(x)|>\rho\}=0\big\} & \text{if } \alpha\neq 0\end{cases}\\ &=|\alpha|\|f\|_\infty.\end{aligned}$$

Thirdly we compute for an arbitrary $\varepsilon>0$

$$\begin{aligned}&\mu(\{x\in X:|f(x)+g(x)|>\|f\|_\infty+\|g\|_\infty+2\varepsilon\}\\ &\quad\le\mu(\{x\in X:|f(x)|>\|f\|_\infty+\varepsilon\}+\mu(\{x\in X:|g(x)|>\|g\|_\infty+\varepsilon\}=0,\end{aligned}$$

which implies that $\|f+g\|_\infty\le 2\varepsilon+\|f\|_\infty+\|g\|_\infty$, and thus, since $\varepsilon>0$ was arbitrary, that $\|f+g\|_\infty\le\|f\|_\infty+\|g\|_\infty$,
(c) Let $r_n=\|f_n-f\|_\infty$ and

$$A=\bigcup_{n=1}^\infty\Big\{x\in X:|f_n(x)-f(x)|\le r_n+\frac{1}{n}\Big\}.$$

Then A is a null set and $f_n(x)$ converges uniformly to $f(x)$ on $X\setminus A$.
(d) Assume that f_n is absolutely converging in $L_\infty(\mu)$. Let

$$A=\bigcup_{n\in\mathbb{N}}\{x\in X:|f_n(x)|\ge 2^n+\|f_n\|_\infty\}.$$

A is a μ-nullset and $f(x)=\sum_{n\in\mathbb{N}}f_n(x)$ exists for all $x\in X\setminus A$.

Moreover for $x\in X\setminus A$,

$$|f(x)|\le 1+\sum_{n\in\mathbb{N}}\|f_n\|$$

which implies that $f\in L_\infty(\mu)$, and, for $n\in\mathbb{N}$ and $x\in X\setminus A$

$$\Big|\sum_{j=n+1}^\infty f_j(x)\Big|\le\sum_{j=n+1}^\infty|f_j(x)|\le 1+\sum_{j=n+1}^\infty\|f_j\|_\infty,$$

which implies that $\sum_{n\in\mathbb{N}}f_n$ converges in $L_\infty(\mu)$ to f.
(e) This essentially follows from the same argument with which we proved that positive measurable functions are limits of step functions. Indeed, let $f\in L_\infty(\mu)$, and define $r=\|f\|_\infty+1$ (the point being that r needs to be slightly bigger than $\|f\|_\infty$). Let $\varepsilon>$ and choose $N\in\mathbb{N}$, with $N>1/\varepsilon$, Then define for $j=1,2,\dots,\lceil 2rN\rceil$

$$A_j=\Big[-r+\frac{j-1}{N},-r+\frac{j}{N}\Big),$$

And then take

$$\tilde f=\sum_{j=1}^{\lceil 2rN\rceil}\frac{j-1}{N}\chi_{f^{-1}(A_j)}.$$

(f). We note that the L_∞- distance of $\chi[0,1]$ to any continuous function is at least 1/2. Indeed, let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function, and assume that $\|f-\chi_{[0,1]}\|_\infty = r < \frac{1}{2}$. Then $f \geq r$ a.e. on $[0,1]$. But this implies by the continuity of f that there is a $\delta > 0$ so that $f(x) > 1/2$ for all $x \in (-\delta, 0)$ (let $\varepsilon = \frac{1}{2} - r$), which means that

$$m(\{x \in (\infty, 0] : f(x) > \frac{1}{2}\}) \neq 0$$

and thus we get a contradiction.

Problem 3. Let $1 < p < \infty$. Find a function $f : [0,1] \to \mathbb{R}$ which is in $\bigcap_{r<p} L_r[0,1]$, but $f \notin L_p[0,1]$ and find a function $g \to \mathbb{R}$ which is in $L_p[0,1]$ but not in $\bigcap_{r>p} L_r[0,1]$.

Proof. Put $f = x^{-1/p}$, $0 < x \leq 1$.

$$\|f\|_p^p = \lim_{\varepsilon\to 0} = \int_\varepsilon^1 \frac{1}{x}\,dx = -\ln\varepsilon \to \infty_{\varepsilon\to 0},$$

Thus $f \notin L_p[0,1]$. But since $f \geq$ and since $x^{r/p}$ is Riemann intergrable for all $r < p$, it follows that $f \in L_r(\mu)$, for $r < p$.

In oder to define g we let (r_n) be decreasing sequence in $(p,1]$ which converges to p, and define for $n \in \mathbb{N}$, $g_n(x) = x^{1/r_n}$, $0 < x \leq 1$. By the above observations it follows that $g_n \in L_p[0,1]$ for all $n \in \mathbb{N}$, and $g_n \notin L_{r_n}[0,1]$, $n \in \mathbb{N}$,

Define

$$g(x) = \sum_{n=1}^\infty 2^{-n} \frac{g_n(x)}{\|g_n\|_p},$$

then the series is absolute convergent in $L_p[0,1]$ and thus convergent, and $\|g\|_p = 1$. But for each $n \in \mathbb{N}$,

$$\|g\|_{r_n} \geq 2^{-n}\|g_n\|_{r_n}/\|g_n\|_p = \infty.$$

Since $L_r[0,1]$ is a subset of $L_{r'}[0,1]$ if $r > r'$, it follows that g cannot be in any $L_r[0,1]$, with $r > p$.

Problem 4. Let $1 \leq p < \infty$. For sequence $(f_n) \subset L_p(\mu)$ ($(X, \mathcal{M}, \mu)$ measure space) and $f \in L_p(\mu)$ it follows that

$$f_n \to_{n\to\infty} f \text{ in } L_p \iff f_n \to f \text{ in measure and } \|f_n\|_p \to_{n\to\infty} \|f\|_p.$$

Proof. "$\Rightarrow$" is easy, by noting that for $\varepsilon > 0$

$$\mu(\{|f_n - f| > \varepsilon\}) = \mu(\{|f_n - f|^p > \varepsilon^p\}) \leq \varepsilon^{-p} \int_{|f_n-f|^p > \varepsilon^p} |f_n - f|\,d\mu \leq \varepsilon^{-p}\|f - f_n\|_p^p.$$

"$\Leftarrow$" There are two proofs. The one "by foot"I came up with, and the other, more elegant proof, which uses the generalized Dominated Concergence Theorem.

: Long Proof

We first note that we can assume that f_n converges μ a.e. to f. Indeed, every subsequence of (f_n) has a further subsequence which which converges a.e. and it is enough to show that every subsequence of (f_n) has a further subsequence which is convergent to f in L_p. It is also enough to show that there is a subsequence of (f_n) which converges in L_p to f, because this would mean that we have shown that every subsequence has a further subsequence which converges in L_p to f.

We first claim that $\lim_{n\to\infty} \|f + f_n\|_p = 2\|f\|_p$. Indeed,

$$\begin{aligned}
2\|f\|_p &= \lim_{n\to\infty} \|f\|_p + \|f\|_p \\
&\geq \liminf_{n\to\infty} \|f + f_n\|_p \text{ (By triangle-inequality)} \\
&= \Big[\liminf_{n\to\infty} \int |f + f_n|^p d\,\mu \Big]^{1/p} \\
&\geq \Big[\int \liminf_{n\to\infty} |f + f_n|^p d\,\mu \Big]^{1/p} \text{ (By Fatou)} \\
&= \Big[\int |2f|^p d\,\mu \Big]^{1/p} = 2\|f\|_p.
\end{aligned}$$

Next we claim that for all $\varepsilon > 0$ there is a $C > 0$ so that

$$\|f_n \chi_{\{|f_n| > C|f|\}}\|_p < \varepsilon, \text{ for all } n \in \mathbb{N}.$$

Assume that where not true, then we could, by passing again to a subsequence, assume w.l.o.g. that there is an ε_0

$$\|f_n \chi_{\{|f_n| > 2^n|f|\}}\|_p \geq \varepsilon_0, \text{ for all } n \in \mathbb{N}.$$

Note that $g_n = f_n \chi_{\{|f_n| > 2^n|f|\}}$ converges in measure to 0, indeed

$$\begin{aligned}
\mu(\{|g_n| > r\}) &\leq \mu(\{|f - f_n|\chi_{\{|f_n| \geq 2^n|f|\}} > r/2\}) + \mu(\{|f|\chi_{\{|f_n| \geq 2^n|f|\}} \geq r/2\}) \\
&\leq \mu(\{|f - f_n| > r/2\}) + \mu(\{|f|\chi_{\{|f_n| \geq 2^n r/2\}} \geq r/2\}) \to_{n\to\infty} 0.
\end{aligned}$$

(Note that $\mu(\{|f_n| \geq 2^n r/2\}) \to_{n\to\infty} 0$). After passing to a subsequence we may assume that g_n converges to 0 a.e..

Thus, as above (using Fatou) we can conclude that for $\tilde{f}_n = f_n - g_n = f_n \chi_{\{|f_n| \leq 2^n|f|\}}$, that

$$\liminf_{n\to\infty} \|f + \tilde{f}_n\|_p \geq 2\|f\|_p.$$

But on the other hand

$$\begin{aligned}
\|f + \tilde{f}_n\|_p &\leq \|f\|_p + \|\tilde{f}_n\|_p \\
&= \|f\|_p + \big[\|f_n\|^p - \|g_n\|_p^p\big]^{1/p} \text{ (Disjointness of supports)} \\
&\leq \|f\|_p + \big[\|f_n\|^p - \varepsilon_0^p\big]^{1/p} \to \|f\|_p + \big[\|f\|^p - \varepsilon_0^p\big]^{1/p} < 2\|f\|_p.
\end{aligned}$$

Now the "endgame" goes as follows. Let $\varepsilon > 0$. Choose $C = C_\varepsilon$ so that $\|f_n \chi_{\{|f_n| > C|f|\}}\|_p < \varepsilon$, for all $n \in \mathbb{N}$.

It follows

$$\begin{aligned}\|f_n - f\|_p^p &= \int \chi_{\{|f_n| \le C|f|\}} |f_n - f|^p \, d\mu + \int \chi_{\{|f_n| > C|f|\}} |f_n - f|^p \, d\mu \\ &\le \int \chi_{\{|f_n| \le C|f|\}} |f_n - f|^p \, d\mu + \int \chi_{\{|f_n| > C|f|\}} 2^p |f_n| \, d\mu + \int \chi_{\{|f_n| > C|f|\}} 2^p |f| \, d\mu \\ &\le \varepsilon + \int \chi_{\{|f_n| \le C|f|\}} |f_n - f|^p \, d\mu + \int \chi_{\{|f_n| > C|f|\}} 2^p |f| \, d\mu \to_{n\to\infty} \varepsilon,\end{aligned}$$

by the Dominated Convergence Theorem. Since $\varepsilon > 0$ was arbitrary, we deduce that $\lim_{n\to\infty} \|f - f_n\|_p = 0$.
Shorter Proof:
First note that as argued above we can assume that f_n is a.e. convergent to f. Then apply the "Generalized Dominated Convergence Theorem" (Page 59 problem 21). To the sequence $(|f - f_n|^p : n \in \mathbb{N})$, noting that

$$\tilde{f}_n := |f - f_n|^p \le (|f| + |f_n|)^p \le 2^p(|f| + |f_n|) =: g_n,$$

and that g_n converges almost every where to the function $g = 2^p|f|$. It follows from the Generalized Dominated Convergence Theorem that $\int \tilde{f}_n dmu \to_{n\to\infty} 0$, which implies our claim.

Problem 5. A linear functional on a normed linear space is bounded if and only if $f^{-1}(\{0\})$ is closed.
Proof. $\Rightarrow$ is clear since bounded linear functionals on a normed linear space are continuous, and the preimages of closed sets under continuous maps are closed.
$\Leftarrow$ W.l.o.g. f is not the zero functional (which is clrealy bounded). So let $x_0 \in X$ with $f(x_0) \neq 0$, say $f(x_0) = 1$ (after multipliction with the right scalar). Then we can wirte every $x \in X$ as $x = f(x)x_0 + y$ with $y \in Y = f^{-1}(\{0\})$ (simply note that $x - f(x)x_0$ must be in $f^{-1}(\{0\})$). Now use Theorem of Hahn-Banach (more precisely Theorem 5.8 (a) on page 159) to get a linear bounded functional g on X so that $g||_Y = 0$ and $g(x_0) \neq 0$. After multiplying g with the right scalar we may assume $g(x_0) = 1$. Now we claim that $g = f$, and thus f must be continious.

Indeed for all $x \in X$ we find $y \in Y$ with $x = f(x)x_0 + y$, and thus

$$g(x) = g(f(x)x_0 + y) = f(x)g(x_0) + g(y) = f(x).$$

Problem 6. Suppose X and Y are normed linear spaces and $T \in L(X, Y)$. Define $T^t : Y^* \to X^*$ by $y^* \mapsto y^* \circ T$ (which must be in X^* since y^* and T are both bounded and linear).

a) $T \in L(Y^*, X^*)$ and $\|T^t\| = \|T\|$.
b) Let $I_X : X \to X^{**}$ and $I_X : Y \to Y^{**}$ be the canonical embeddings, then $T^{tt} \circ I_X = I_Y \circ T$.
c) T^t injective $\iff$ $\overline{T(X)} = Y$.
d) If the range of T^t is dense in X^*, then T is injective. If X is reflexive the converse holds.

Proof. (a) Clearly T^t is welldefined and linear. Moreover

$$\begin{aligned}\|T^t\| &= \sup_{f\in Y^*,\|f\|\le 1} \|f\circ T\|_{X^*}\\ &= \sup_{f\in Y^*,\|f\|\le 1}\ \sup_{x\in X,\|x\|\le 1} |f\circ T(x)|\\ &\le \sup_{f\in Y^*,\|f\|\le 1}\ \sup_{x\in X,\|x\|\le 1} \|f\|\cdot\|T\|\cdot\|x\| = \|T\|.\end{aligned}$$

On the other hand let $\varepsilon > 0$. Choose $x \in X$, $\|x\| = 1$ so that $\|T(x)\| \ge \|T\|(1-\varepsilon)$. Secondly choose by Hahn Banach $f \in Y^*$, $\|f\| = 1$ so that $f(T(x)) = \|T(x)\|$. Thus

$$\|T^t\| \ge \|T^t(f)\| \ge |T^t(f)(x)| = f(T(x)) = \|T(x)\| \ge \|T\|(1-\varepsilon),$$

which implies the claim since $\varepsilon >$ was asumed to be arbitrary.
(b) Let $x \in X$. To show $T^{tt}\circ I_X(x) = I_Y\circ T(x)$. This is equivalent to show that for all $y^* \in Y^*$ it follows that $\big[T^{tt}\circ I_X(x)\big](y^*) = \big[I_Y\circ T(x)\big](y^*)$.

Then note that for $x \in X$ and $y^* \in Y^*$ it follows that

$$\begin{aligned}\big[T^{tt}\circ I_X(x)\big](y^*) &= I_X(x)(T^*(y^*)) \text{ (By Definition of } T^{tt})\\ &= T^*(y^*)(x) \text{ (By Definition of } I_X)\\ &= y^*(T(x)) \text{ (By Definition of } T^t)\\ &= I_Y(T(y))(y^*) \text{ (By Definition of } I_Y\end{aligned}$$

(c) *not* $\Leftarrow$ *not*: Let $y_0 \in Y\setminus\overline{T(X)}$. By the corollary of the Hahn Banach theorem we find an $f \in Y^*$ with $f(y_0) = 1$ and $f|_{\overline{T(X)}} = 0$. Thus for all $x \in X$ it follows that $T^t(f)(x) = f(T(x)) = 0$, thus $T^t(f) = 0$, which means that T^t is not injective.
(c) *not* $\Rightarrow$ *not*: Assume that $f \in Y^*\setminus\{0\}$ and $T^t(f) = 0$. This means that for all $x \in X$ it follows that $0 = T^t(f)(x) = f(T(x))$. But this means that $\mathrm{Rg}(T) \subset \mathrm{Ker(f)}$ and thus $\overline{\mathrm{Rg}(T)} \subset \mathrm{Ker(f)} \ne Y$.
(d) We prove *not* $\Leftarrow$ *not* and assume that $T(x) = 0$ for some $x \in X\setminus\{0\}$. It follows for all $y^* \in Y^*$, that $T^t(y^*)(x) = y^*(T(x)) = 0$. This means that $rg(T) \subset \{x^* \in X^* : \hat{x}(x^*) = x^*(x) = 0\} = A$. Now A is closed By Hahn Banach it follows that there is an $f \in X^*$ with $f(x) \ne 0$, thus $f \notin A$ and therefore A, and thus $\overline{\mathrm{rg}(T)}$ can not be equal to Y.

If X is reflexive the converse follows from part (b) and (c).

Problem 7. Problem 18/page 159 Let X be a normed linear space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

a) If Y is a closed subspace and $x \in X\setminus Y$ then $x\mathbb{F}+Y$ is closed
b) Every finite dimensional subspace of X is closed.

Proof. (a) Let $z_n = \lambda_n x + y_n$ be in $x\mathbb{F}+Y$, and assume that z_n converges in X to some z. We have to show that $z \in x\mathbb{F}+Y$.

Apply the projection P onto the quotient space Y/X (see Problem 1 Homework 4). Then $P(z_n) = P(\lambda_n x) = \lambda_n P(x)$ converges to $P(z)$ (by continuity of P). Since $P(x) \neq 0$, it follows that

$$\lim_{n\to\infty} |\lambda_n| = \|P(z)\|/\|P(x)\|,$$

and thus that λ_n has a subsequence (λ_{n_j}) which converges to some $\lambda \in \mathbb{F}$. Thus $y_{n_j} = z_{n_j} - \lambda_{n_j} x$ converges to $z - \lambda x$ which therefore has to be in Y (Y is closed), thus for some $y \in Y$, $z = \lambda x + y$.

(b) follows by induction on the dimension of Y, in each step using (a).

Problem 8. Problem 19/page 160 Let X be an infinite-dimensional normed vector space.

a) There is a sequence $(x_n) \subset X$, with $\|x_n\| = 1$, for all $j \in \mathbb{N}$, and $\|x_j - x_i\| \geq \frac{1}{2}$, if $i \neq j$.
b) The unit ball $B_X = \{x \in X, \|x\| \leq 1\}$ is not compact.

Remark. A much deeper Theorem by Dor and Odell the following is true: For very infinite dimensional Banach space X there is a an $\varepsilon = \varepsilon_X > 0$, so that there is a sequence $(x_j) \in S_X = \{x \in X : \|x\| = 1\}$ so that $\|x_i - x_j\| \geq 1 + \varepsilon$, if $i \neq j$.

See if you can proof the following weaker statement: There is a sequence $(x_j) \subset S_X$ so that $\|x_i - x_j\| > 1$, if $i \neq j$. (this is not a required homework).

Proof. By induction we choose $x_1, x_2, \ldots$ in S_X so that $\|x_j - x_i\| \geq 1/2$ whenever $i \neq j$.

Choose x_1 arbitrary and assume that $x_1, \ldots x_n$ have been chosen. Let Y be the space generated by $x_1, \ldots x_n$. By Problem 7 b, Y is a closed subspace of X.

By Problem 1b/Homework 4, there is an x_{n+1} so that $\|x_{n+1}\| = 1$ and $\|x_{n+1} + Y\| > 1/2$. Thus for all $i = 1, 2 \ldots n$,

$$\|x_{n+1} - x_i\| \geq \operatorname{dist}(x_{n+1}, Y) = \|x_{n+1} + Y\| > \frac{1}{2}.$$