Problems to Introduction to Real Analysis, (Math446)

Due: 11/11/04

Problem 1. Assume that (X, d_X), (Y, d_Y) and (Z, d_Z) are metric spaces, and assume that $f : X \to Y$ and $g : Y \to Z$ are continuous. Show that $g \circ f$ is continuous.

Problem 2. Prove that addition $+ : \mathbb{R}^2 \to \mathbb{R}$ and multiplication $\cdot : \mathbb{R}^2 \to \mathbb{R}$ are continuous.

Problem 3. Denote by $\mathbb{R}^{\mathbb{N}}$ the set of all sequences in $\mathbb{R}$. For (x_n) and (y_n) in $\mathbb{R}^{\mathbb{N}}$ put

$$d((x_n), (y_n)) = \sum_{n\in\mathbb{N}} 2^{-n} \min(|x_n - y_n|, 1).$$

Sow that $(\mathbb{R}^{\mathbb{N}}, d)$ is a complete metric space and that for a sequence $(x^{(n)})_{n=1}^{\infty}$ (i.e. $x^(n) = (x_m^{(n)})_{m=1}^{\infty} \subset \mathbb{R}$ for each $n \in \mathbb{N}$) it follows that $(x^(n))_{n=1}^{\infty}$ converges to some $x = (x_m) \in \mathbb{R}^{\mathbb{N}}$ if and only if for all $m \in \mathbb{N}$ $(x_m^{(n)})_{n=1}^{\infty}$ converges to x_m (in $\mathbb{R}$).

Problem 4. Assume that (X, d) is a complete metric space, and that $\mathcal{F}$ is a set of $\mathbb{R}$ valued continuous functions on X and assume that for each $x \in X$

$$\sup_{f\in\mathcal{F}} |f(x)| < \infty.$$

Show that there is an open and not empty set $U \subset X$ so that

$$\sup_{x\in U} \sup_{f\in\mathcal{F}} |f(x)| < \infty.$$

Problem 5. Assume that U is an open and F is a closed subset of a metric space (X, d) show that $\overline{U} \setminus U$ and $F \setminus F^\circ$ are of first category.

Find a set $A \subset \mathbb{R}$ so that $A \setminus A^\circ$ is not of first category.

Problem 6. Consider the following topology on the set $X = \{a, b, c, d, e\}$

$$\mathcal{T} = \big\{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\big\}.$$

a) List all closed sets
b) Determine the closure of $\{a\}$, $\{b\}$ and $\{c, e\}$,
c) Which of the sets in (b) are dense?