

Prof.: Thomas Schlumprecht

For Problems 1 and 2 Recall: $\ell_\infty = \{x = (x_i)_{i \in \mathbb{N}} \subset \mathbb{F} : \|x\|_\infty = \sup |x_i| < \infty\}$,

$$\ell_1 = \{x = (x_i)_{i \in \mathbb{N}} \subset \mathbb{F} : \|x\|_1 = \sum |x_i| < \infty\}, \text{ and}$$

$$c_0 = \{x = (x_i)_{i \in \mathbb{N}} \subset \mathbb{F} : \lim_{n \rightarrow \infty} x_n = 0\}.$$

Problem 1. Show that there is an $f \in \ell_\infty^*$ so that $\|f\| = 1$, $f(1, 1, 1, \dots) = 1$ and $f|_{c_0} \equiv 0$.

Problem 2. Show that c_0^* and ℓ_1 are isometrically isomorphic, via the "scalar product", i.e an element $y = (y_n) \in \ell_1$ acts on an element $x = (x_n) \in c_0$ by

$$y(x) = \langle y, x \rangle = \sum_{n \in \mathbb{N}} y_n x_n.$$

Problem 3. Problem 24/page 160. Suppose that X is a Banach space. Let \widehat{X} and $(X^*)^\wedge$ be the images of X and X^* in X^{**} and X^{***} , respectively under the canonical maps

$$\varkappa : X \hookrightarrow X^{**} \text{ and } \overline{\varkappa} : X^* \hookrightarrow X^{***}.$$

Define

$$\widehat{X}^0 = \{F \in X^{***} : F|_{\widehat{X}} \equiv 0\}.$$

- Show that $\widehat{X}^0 \cap (X^*)^\wedge = \{0\}$ and $\widehat{X}^0 + (X^*)^\wedge = X^{***}$.
- Show that X is reflexive if and only if X^* is reflexive.

Problem 4. Show that every weakly converging sequence in a normed linear space is bounded.

Problem 5. Show that every infinite dimensional Banach space admits nets $(x_i)_{i \in I}$ which converge weakly to 0, but are "cofinally unbounded", which means that for any $i \in I$ and any $C > 0$ there is a $j \in I$, $j \geq i$ so that $\|x_j\| \geq C$.

Hint: For the second part it might be useful to first prove that for an infinite dimensional Banach X and functionals $f_1, f_2, \dots, f_n \in X^*$ the intersection of the null sets of the f_i 's contains non zero elements. This can be deduced using basics in linear algebra.

Problem 6. Show that if (x_n) converges weakly in a Banach space X to some $x \in X$ then

$$\liminf_{n \rightarrow \infty} \|x_n\| \geq \|x\|.$$

Give an example which shows that the inequality can be strict.

Problem 7. Show that f_n , with

$$f_n : [0, 1] \rightarrow \mathbb{R}, \quad x \mapsto \sin(2\pi n x),$$

is weakly converging to 0 in $L_p[0, 1]$, $1 < p < \infty$.