Problems in Real Variables, II (Math608), Solutions
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Problem 1. Show that there is an $f \in \ell_\infty^*$ so that $\|f\| = 1$, $f(1,1,1,\ldots) = 1$ and $f|_{c_0} \equiv 0$.

Proof. Note that the distance between the subspace c_0 and the vector $(1,1,\ldots)$ in ℓ_∞ is 1 i.e.

$$\text{dist}(c_0,(1,1,\ldots)) = \inf_{y\in c_0} \|(1,1,\ldots) - y\|_\infty = \inf_{y\in c_0} \sup_{n\in\mathbb{N}} |y_n - 1| = 1.$$

Using a theorem shown in class (Theorem 5.8 (a) on page 159) it follows that there must be a bounded functional $f \in \ell_\infty$ so that $f|_{c_0}$, $f((1,1,\ldots..)) = 1$ and $\|f\|_{\ell_\infty^*} = 1$.

Problem 2. Show that c_0^* and ℓ_1 are isometrically isomorphic, via the "scalar product", i.e an element $y = (y_n) \in \ell_1$ acts on an element $x = (x_n) \in c_0$ by

$$y(x) = \langle y, x\rangle = \sum_{n\in\mathbb{N}} y_n x_n.$$

$$\begin{aligned} \text{Recall:}\ell_\infty &= \{x = (x_i)_{i\in\mathbb{N}} \subset \mathcal{F} : \|x\|_\infty = \sup |x_i| < \infty\} \\ \ell_1 &= \{x = (x_i)_{i\in\mathbb{N}} \subset \mathcal{F} : \|x\|_1 = \sum |x_i| < \infty\} \\ c_0 &= \{x = (x_i)_{i\in\mathbb{N}} \subset \mathcal{F} : \lim_{n\to\infty} x_n = 0\} \end{aligned}$$

Proof. Consider the following mapping:

$$I : \ell_\infty \to \ell_1^*, \quad (x_i) \mapsto I((x_i)), \text{ with } I((x_i))(z_i) = \sum_{i\in\mathbb{N}} z_i x_i$$

(a) I is well defined, i.e. $I(x) \in \ell_1^*$ for $x = (x_i) \in \ell_\infty$.
Indeed for $z = (z_i) \in \ell_1$, it follows that

$$(*) \qquad \sum_{i\in\mathbb{N}} |z_i x_i| \le \sup_{i\in\mathbb{N}} |z_i| \sum_{i\in\mathbb{N}} |z_i| = \|x\|_\infty \|z\|_1.$$

Thus $\sum_{i\in\mathbb{N}} z_i x_i$ is convergent, and it is easy to see that $I(x)$ is a linear, map on ℓ_1. Thus by (*) it follows that

$$\|I(x)\|_{\ell_1^*} = \sup_{z=(z_i)\in\ell_1, \|x\|_1\le 1} |\sum_{i\in\mathbb{N}} z_i x_i| \le \|x\|_\infty.$$

(b) I is an isomteric injection.
Clearly $I : \ell_\infty \to \ell_1^*$ is linear, and from $(*)$ it follows that I is bounded and that $\|I\| \le 1$. Still to show: for an $x \in \ell_\infty$ it follows that $\|I(x)\|_{\ell_1^*} \ge \|x\|_\infty$. If $x = (x_n) \in \ell_\infty$ and $\varepsilon > 0$ we find an $n \in \mathbb{N}$ so that $|x_n| \ge \sup_{i\in\mathbb{N}} |x_i| - \varepsilon = \|x\|_\infty - \varepsilon$. Let e_n to be the "n-th" unit vector in ℓ_1 (i.e. $e_n = (0,0,\ldots,0,1,0\ldots)$ with the "1" being in the n-coordinate). Choose $z = \text{sign}(x_n)e_n$ (whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) and then. $I(x)(z) = |x_n| \ge \|x\|_\infty - \varepsilon$. Since $\|z\| = 1$ this implies that $\|I(x)\| \ge 1 - \varepsilon$ and since $\varepsilon > 0$ is arbitrary the claim follows.

(c) Surjectivity of I. Let $f \in \ell_1^*$. Define $x_i = f(e_i)$ for $i \in \mathbb{N}$. First note that $\sup |x_i| = \sup |f(e_i)| \le \|f\|_{\ell_1^*}$. Secondly for $x = (x_i)$ and $z \in \ell_1$:

$$I(x)(z) = \sum_{i\in\mathbb{N}} x_i z_i = \lim_{N\to\infty} \sum_{i=1}^{N} z_i f(e_i) = \lim_{N\to\infty} f(\sum_{i=1}^{N} z_i e_i) = \lim_{N\to\infty} f(z^{(N)}) = f(z)$$

where $z^{(N)} = \sum_{i=1}^{N} z_i e_i$ and where the last $=$ follows from the fact that $Z^{(N)}$ converges in ℓ_1 to z.

Problem 3. Problem 24/page 160. Suppose that X is a Banach space. Let $\widehat{X}$ and $(X^*)^\wedge$ be the images of of X and X^* in X^{**} and X^{***}, respectively under the canonical maps

$$\varkappa : X \hookrightarrow X^{**} \text{ and } \overline{\varkappa} : X^{*}{*} \hookrightarrow X^{***}.$$

Define

$$\widehat{X}^0 = \{F \in X^{***} : F|_{\widehat{X}} \equiv 0.$$

a) Show that $\widehat{X}^0 \cap (X^*)^\wedge = \{0\}$ and $\widehat{X}^0 + (X^*)^\wedge = X^{***}$.
b) Show that X is reflexive if and only if X^* is reflexive.

Proof.
(a) If $F \in \widehat{X}^0 \cap (X^*)^\wedge$ then we can write $F = \overline{\varkappa}(f)$, with $f \in X^*$ and we conclude that for any $x \in X$

$$\begin{aligned}\langle f, x\rangle &= \langle \varkappa(x), f\rangle \\ &= \langle \overline{\varkappa}(f), \varkappa(x)\rangle \\ &= \langle F, \varkappa(x)\rangle = 0 \text{ since } F \in \widehat{X}^0,\end{aligned}$$

which implies that f and, thus $F = \overline{\varkappa}(f)$ is 0.

Let $F \in X^{***}$ be arbitrary. Define $f \in X^*$ by

$$f : X \to \mathbb{F}, \quad x \mapsto F(\varkappa(x)) = \langle F, \varkappa(x)\rangle.$$

It is easy to see that f is linear and bounded and thus in X^*. Then we can write

$$F = \overline{\varkappa}(f) + (F - \overline{\varkappa}(f)).$$

We still need to show that $(F - \overline{\varkappa}(f)) \in \widehat{X}^0$ But this follows from the fact that for any $x \in X$ we have

$$\langle F, \varkappa(x)\rangle = \langle f, x\rangle = \langle \varkappa(x), f\rangle = \langle \overline{\varkappa}(f), \varkappa(x)\rangle.$$

(b) From part (a) we deduce

$$\begin{aligned}X \text{ reflexive } &\Rightarrow \widehat{X} = \{0\} \\ &\Rightarrow X^{***} = (X^*)^\wedge \Rightarrow X^* \text{ reflexive}\end{aligned}$$

Conversely if X is not reflexive and thus $\widehat{X} \subsetneq X^{**}$, we can find by the Hahn Banach theorem an $F \in X^{***}$ which vanishes on $\widehat{X}$ but $F \neq 0$. Thus, $\widehat{X}^\wedge \neq 0$, and again by part (a) $(X^*)^\wedge \neq X^{***}$, which means that X^* is not reflexive.

Problem 4. Show that every weakly converging sequence in a normed linear space is bounded. **Proof.** In order to show the first part we assume that (x_n) is an unbounded sequence which converges to some x. After subtracting x we may assme that x_n converges to 0.

By induction we choose a subsequence (x_{n_k}) and functionals (x_k^*) so that for all $k \in \mathbb{N}$:

$$x_k^*(x_{n_k}) \geq 2^k, \tag{1}$$

$$|x_k^*(x_{n_j})| \leq 2^{-k} \text{ if } j < k, \tag{2}$$

$$|x_j^*(x_{n_k})| \leq 2^{-k} \text{ if } j < k, \text{ anf} \tag{3}$$

$$\|x_k^*\| \leq 2{-}k. \tag{4}$$

For $k = 1$ we choose n_1 so that $\|x_{n_1}\| \geq 2$ and then we use the Hahn Banach Theorem to find an $x_1^* \in X^*$, $\|x_1^*\| = 1$ so that $x_1^*(x_{n_1}) \geq \|x_{n_1}\|$.

Assume we have chosen $n_1 < n_2 \dots n_k$ in N. We first choose $m > n_k$ so that for all $j = 1, 2 \dots k$ we have $|x_j^*(x_m)| \leq 2^{k+1}$ (using that (x_n) is weakly null). Then, using the unboundedness of (x_n) we choose $n_{k+1} \geq m$ so that

$$\|x_{n_{k+1}}\| \geq (1 + \max_{j=1,2\dots k} \|x_{n_j}\|)2^{2k+2},$$

and use again the Hahn Banach Theorem to find an $x^* \in X^*$, $\|x^*\| = 1$, so that $x^*(x_1) \geq (1 + \max_{j=1,2\dots k} \|x_{n_j}\|)2^{2k+2}$ then let $x_{k+1}^* = x^*/((1 + \max_{j=1,2\dots k} \|x_{n_j}\|)2^{k+1})$. It is now easy to check that all the wanted conditions are satisfied. This finishes the induction step.

Form condition (4) and the completeness of X^* it follows that $x^* = \sum x_k^*$ converges and $\|x^* \leq 1$.

It follows secondly from (1) (2) and (3) for each $k \in \mathbb{N}$ that

$$\langle x^*, x_{n_k}\rangle \geq \langle x^*, x_{n_k}\rangle - \sum_{j<k} |\langle x_j^*, x_{n_k}\rangle| - \sum_{j>k} |\langle x_j^*, x_{n_k}\rangle| \geq 2^k - k2^{-k} - 1 \to_{k\to\infty} \infty$$

which is a contradiction to the weak convergence of (x_n).

Problem 5. Show that every infinite dimensional Banach space admits nets $(x_i)_{i\in N}$ which converge to 0, but are "cofinally unbounded", which means that for any $i \in I$ and any $C > 0$ there is a $j \in I$, $j \geq i$ so that $\|x_j\| \geq C$.

Hint: For the second part it might be useful to first prove that for an infinite dimensional Banach X and functionals $f_1, f_2, \dots f_n \in X^*$ the intersection of the null sets of the f_i's contains non zero elements. This can be deduced using basics in linear algebra.

Proof. We will first show the hint and use the fact from linear algebra, that for a linear transformation $T : V \to W$ between two finite dimensional vectorspaces it follows that

$$\dim(V) = \dim \operatorname{range}(T) + \dim \mathcal{N}(T),$$

where $\mathcal{N}(T)$ denotes the null space of T.

Now let Y be any $n+1$ dimensional subspace of X and let $f_i|_Y$ be the restriction of the f_i's to Y.

Since $\dim(\text{range}(f_1) \le 1$ it follows that $\dim(\mathcal{N}(f_1|_Y) \ge n$. Now we consider $f_2|_{Y\cap\mathcal{N}(f_1)}$ and deduce with the same argument

$$\dim(\mathcal{N}(f_1|_Y)\cap\mathcal{N}(f_2|_Y)) = \dim(\mathcal{N}(f_2|_{Y\cap\mathcal{N}(f_1)})) \ge \dim(\mathcal{N}(f_1)\cap Y)-1 \ge n-1.$$

We can continue with this argument and obtain after n steps that

$$\dim\Big(\bigcap_{i=1}^n \mathcal{N}(f_i|_Y)\Big) \ge 1.$$

Now the construction of our Net. We define as our index set, the typical neighborhood basis os 0 in X, i.e.

$$\mathcal{I} = \Big\{\bigcap_{i=1}^n \{x\in X : |f_i(x)| \le \varepsilon\} : n\in\mathbb{N}, \varepsilon>0, f_i\in X^*, \|f_i\|=1, \text{ for } i=1,\dots n\Big\}.$$

The order we consider is, as usual, the reversed inclusion.

For each $U = \{x\in X : |f_i(x)| \le \varepsilon\} \in I$ we pick an $x_U \in \bigcap_{i=1}^n \mathcal{N}(f_i)$ with $\|x_U\| \ge 1/\varepsilon$. Since for every $U\in\mathcal{I}$, $x_U\in U$, and $\mathcal{I}$ is a neighborhood basis of 0, it follows that $(x_U)_{U\in\mathcal{I}}$ is converging to 0, and it is clearly cofinally unbounded (simply make make ε small enough).

Problem 6. Show that if (x_n) converges weakly in a Banach space X to some $x\in X$ then

$$\liminf_{n\to\infty} \|x_n\| \ge \|x\|.$$

Give an example which shows that the inequality can be strict.

Proof. Assume that x_n converges weakly to x. By the Theorem of Hahn Banach we can find an $x^*\in X^*$ with $\|x^*\|=1$ and $\langle x^*, x\rangle = \|x\|$. Thus

$$\liminf_{n\to\infty} \|x_n\| \ge \liminf_{n\to\infty} \langle x^*, x_n\rangle = \langle x^*, x\rangle = \|x\|.$$

As example take the unit vector basis (e_n) in c_0, i.e. as above $e_n = (0,0,\dots 0,1,0,\dots)$ with the "1" being in the n coordinate. By representing c_0^* as ℓ_1 (see Problem 2) it follows easily that (e_n) converges weakly to 0, but of course $\|e_n\|=1$, for all $n\in\mathbb{N}$.

Problem 7. Show that f_n, with

$$f_n : [0,1] \to \mathbb{R}, \quad x \mapsto \sin(2\pi n x),$$

is weakly converging to 0 in $L_p[0,1]$, $1<p<\infty$.

Proof (Sketch). Using good old Calculus we observe that for all intervals $I=[a,b]\subset[0,1]$

$$\lim_{n\to\infty} \langle f_n, \chi_I\rangle = \lim_{n\to\infty} \int_a^b \sin(2\pi n x)dx = 0.$$

The prove that the set

$$\mathcal{D} = \Big\{A\subset[0,1]\text{mbl.} : \lim_{n\to\infty} \int_A \sin(2\pi n x)dx = 0\Big\}$$

is Dynkin system (standard!). Thus, by Dynkin's Theorem $\mathcal{D}$ are all measurable subsets of $[0,1]$ and it follows that

$$\lim_{n\to\infty} \int g(x)\sin(2\pi n x)dx = 0,$$

for all simple functions g. Now let $F \in L_p^*[0,1]$. By the representation theorem for $L_p^*[0,1]$ there is an $f \in L_q[0,1]$, $\frac{1}{p} + \frac{1}{q} = 1$, so that

$$F(h) = \int_0^1 f(x)h(x)dx, \text{ for all } h \in L_p[0,1].$$

Now let $\varepsilon > 0$ arbitrary. Since the simple functions are dense in $L_q[0,1]$, we can choose a simple function $\tilde{f}$ with $\|f - \tilde{f}\|_q < \varepsilon$, and thus (note that $|f_n\|_p \le 1$

$$\limsup_{n\to\infty} |\langle F, f_n\rangle| = \limsup_{n\to\infty} \Big| \int f(x)f_n(x)dx\Big| \le \limsup_{n\to\infty} \Big[\Big| \int \tilde{f}(x)f_n(x)dx\Big| + \|f_n\|_p\|f - \tilde{f}\|_q\Big] \le \varepsilon,$$

which implies the claim since $\varepsilon > 0$ was arbitrary.