Problem 1. Show that there is an \( f \in \ell^*_\infty \) so that \( \|f\| = 1 \), \( f(1,1,1,\ldots) = 1 \) and \( f|_{c_0} \equiv 0 \).

**Proof.** Note that the distance between the subspace \( c_0 \) and the vector \((1,1,\ldots)\) in \( \ell_\infty \) is 1 i.e.
\[
\text{dist}(c_0,(1,1,\ldots)) = \inf_{y \in c_0} \|(1,1,\ldots) - y\|_\infty = \inf_{y \in c_0} \sup_{n \in \mathbb{N}} |y_n - 1| = 1.
\]
Using a theorem shown in class (Theorem 5.8 (a) on page 159) it follows that there must be a bounded functional \( f \in \ell_\infty \) so that \( f|_{c_0}, f((1,1,\ldots)) = 1 \) and \( \|f\|_{\ell_\infty} = 1 \).

**Problem 2.** Show that \( c_0^* \) and \( \ell_1 \) are isometrically isomorphic, via the "scalar product", i.e an element \( y = (y_n) \in \ell_1 \) acts on an element \( x = (x_n) \in c_0 \) by
\[
y(x) = \langle y, x \rangle = \sum_{n \in \mathbb{N}} y_n x_n.
\]

Recall:\( \ell_\infty = \{ x = (x_i)_{i \in \mathbb{N}} \subset \mathcal{F} : \|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i| < \infty \} \)
\( \ell_1 = \{ x = (x_i)_{i \in \mathbb{N}} \subset \mathcal{F} : \|x\|_1 = \sum_{i \in \mathbb{N}} |x_i| < \infty \} \)
\( c_0 = \{ x = (x_i)_{i \in \mathbb{N}} \subset \mathcal{F} : \lim_{n \to \infty} x_n = 0 \} \)

**Proof.** Consider the following mapping:
\[
I : \ell_\infty \to \ell_1^*, \quad (x_i) \mapsto I((x_i)), \text{ with } I((x_i))(z_i) = \sum_{i \in \mathbb{N}} z_i x_i
\]

(a) \( I \) is well defined, i.e. \( I(x) \in \ell_1^* \) for \( x = (x_i) \in \ell_\infty \).
Indeed for \( z = (z_i) \in \ell_1 \), it follows that
\[
(*) \quad \sum_{i \in \mathbb{N}} |z_i x_i| \leq \sup_{i \in \mathbb{N}} |z_i| \sum_{i \in \mathbb{N}} |x_i| = \|x\|_\infty \|z\|_1.
\]
Thus \( \sum_{i \in \mathbb{N}} z_i x_i \) is convergent, and it is easy to see that \( I(x) \) is a linear, map on \( \ell_1 \). Thus by \((*) \) it follows that
\[
\|I(x)\|_{\ell_1^*} = \sup_{z=(z_i) \in \ell_1, \|z\|_1 \leq 1} |\sum_{i \in \mathbb{N}} z_i x_i| \leq \|x\|_\infty.
\]

(b) \( I \) is an isometric injection.
Clearly \( I : \ell_\infty \to \ell_1^* \) is linear, and from \((*) \) it follows that \( I \) is bounded and that \( \|I\| \leq 1 \).
Still to show: for an \( x \in \ell_\infty \) it follows that \( \|I(x)\|_{\ell_1^*} \geq \|x\|_\infty \).
If \( x = (x_n) \in \ell_\infty \) and \( \varepsilon > 0 \) we find an \( n \in \mathbb{N} \) so that \( |x_n| \geq \sup_{i \in \mathbb{N}} |x_i| - \varepsilon = \|x\|_\infty - \varepsilon \). Let \( e_n \) to be the “n-th” unit vector in \( \ell_1 \) (i.e. \( e_n = (0,0,\ldots,0,1,0\ldots) \) with the “1” being in the n-coordinate).
Choose \( z = \text{sign}(x_n)e_n \) (whether \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{C} \) and then.
\( I(x)(z) = |x_n| \geq \|x\|_\infty - \varepsilon \).
Since \( \|z\| = 1 \) this implies that \( \|I(x)\| \geq 1 - \varepsilon \) and since \( \varepsilon > 0 \) is arbitrary the claim follows.
(c) Surjectivity of $I$. Let $f \in \ell^*_1$. Define $x_i = f(e_i)$ for $i \in \mathbb{N}$. First note that $\sup |x_i| = \sup |f(e_i)| \leq \|f\|_{\ell^*_1}$. Secondly for $x = (x_i)$ and $z \in \ell_1$:

$$I(x)(z) = \sum_{i \in \mathbb{N}} x_iz_i = \lim_{N \to \infty} \sum_{i=1}^{N} z_i f(e_i) = \lim_{N \to \infty} f(\sum_{i=1}^{N} z_i e_i) = \lim_{N \to \infty} f(z^{(N)}) = f(z)$$

where $z^{(N)} = \sum_{i=1}^{N} z_i e_i$ and where the last $\Rightarrow$ follows from the fact that $Z^{(N)}$ converges in $\ell_1$ to $z$.

**Problem 3.** Problem 24/page 160. Suppose that $X$ is a Banach space. Let $\hat{X}$ and $(X^*)^\wedge$ be the images of of $X$ and $X^*$ in $X^{**}$ and $X^{***}$, respectively under the canonical maps

$$\kappa : X \hookrightarrow X^{**}$$

and

$$\kappa : X^* \hookrightarrow X^{***}.$$ 

Define

$$\hat{X}^0 = \{F \in X^{***} : F|_{\hat{X}} \equiv 0\}.$$ 

a) Show that $\hat{X}^0 \cap (X^*)^\wedge = \{0\}$ and $\hat{X}^0 + (X^*)^\wedge = X^{***}$.

b) Show that $X$ is reflexive if and only if $X^*$ is reflexive.

**Proof.**

(a) If $F \in \hat{X}^0 \cap (X^*)^\wedge$ then we can write $F = \overline{\kappa}(f)$, with $f \in X^*$ and we conclude that for any $x \in X$

$$\langle f, x \rangle = \langle \hat{\kappa}(x), f \rangle$$

$$= \langle \overline{\kappa}(f), \hat{\kappa}(x) \rangle$$

$$= \langle F, \hat{\kappa}(x) \rangle = 0$$

since $F \in \hat{X}^0$,

which implies that $f$ and, thus $F = \overline{\kappa}(f)$ is 0.

Let $F \in X^{***}$ be arbitrary. Define $f \in X^*$ by

$$f : X \to \mathbb{F}, \quad x \mapsto F(\hat{\kappa}(x)) = \langle F, \hat{\kappa}(x) \rangle.$$ 

It is easy to see that $f$ is linear and bounded and thus in $X^*$. Then we can write

$$F = \overline{\kappa}(f) + (F - \overline{\kappa}(f)).$$

We still need to show that $(F - \overline{\kappa}(f)) \in \hat{X}^0$ But this follows from the fact that for any $x \in X$ we have

$$\langle F, \hat{\kappa}(x) \rangle = \langle f, x \rangle = \langle \hat{\kappa}(x), f \rangle = \langle \overline{\kappa}(f), \hat{\kappa}(x) \rangle.$$ 

(b) From part (a) we deduce

$$X$$

reflexive $\Rightarrow \hat{X} = \{0\}$

$\Rightarrow X^{***} = (X^*)^\wedge \Rightarrow X^*$ reflexive

Conversely if $X$ is not reflexive and thus $\hat{X} \nsubseteq X^{**}$, we can find by the Hahn Banach theorem an $F \in X^{***}$ which vanishes on $\hat{X}$ but $F \neq 0$. Thus, $\hat{X}^\wedge \neq 0$, and again by part (a) $(X^*)^\wedge \neq X^{***}$, which means that $X^*$ is not reflexive.
Problem 4. Show that every weakly converging sequence in a normed linear space is bounded. **Proof.** In order to show the first part we assume that \((x_n)\) is an unbounded sequence which converges to some \(x\). After subtracting \(x\) we may assume that \(x_n\) converges to 0.

By induction we choose a subsequence \((x_{n_k})\) and functionals \((x^*_{k})\) so that for all \(k \in \mathbb{N}\):

1. \(x^*_k(x_{n_k}) \geq 2^k\),
2. \(|x^*_k(x_{n_j})| \leq 2^{-k} \text{ if } j < k\),
3. \(|x^*_j(x_{n_k})| \leq 2^{-k} \text{ if } j < k, \text{ and}\)
4. \(|x^*_k\| \leq 2 - k\).

For \(k = 1\) we choose \(n_1\) so that \(|x_{n_1}| \geq 2\) and then we use the Hahn Banach Theorem to find an \(x^*_1 \in X^*, \|x^*_1\| = 1\) so that \(x^*_1(x_{n_1}) \geq \|x_{n_1}\|\).

Assume we have chosen \(n_1 < n_2 \ldots n_k\) in \(N\). We first choose \(m > n_k\) so that for all \(j = 1, 2 \ldots k\) we have \(|x^*_j(x_m)| \leq 2^{k + 1}\) (using that \((x_n)\) is weakly null). Then, using the unboundedness of \((x_n)\) we choose \(n_{k+1} \geq m\) so that

\[
\|x_{n_{k+1}}\| \geq (1 + \max_{j=1,2 \ldots k} \|x_{n_j}\|)2^{2k+2},
\]

and use again the Hahn Banach Theorem to find an \(x^* \in X^*, \|x^*\| = 1\), so that \(x^*(x_1) \geq (1 + \max_{j=1,2 \ldots k} \|x_{n_j}\|)2^{2k+2}\) then let \(x^*_{k+1} = x^*/(1 + \max_{j=1,2 \ldots k} \|x_{n_j}\|)2^{k+1}\). It is now easy to check that all the wanted conditions are satisfied. This finishes the induction step.

Form condition (4) and the completeness of \(X^*\) it follows that \(x^* = \sum x^*_k\) converges and \(|x^*| \leq 1\).

It follows secondly from (1) (2) and (3) for each \(k \in \mathbb{N}\) that

\[
\langle x^*, x_{n_k} \rangle \geq \langle x^*, x_{n_k} \rangle \sum_{j<k} |\langle x^*_j, x_{n_k} \rangle| - \sum_{j>k} |\langle x^*_j, x_{n_k} \rangle| \geq 2^k - k2^{-k} - 1 \to_{k \to \infty} \infty,
\]

which is a contradiction to the weak convergence of \((x_n)\).

Problem 5. Show that every infinite dimensional Banach space admits nets \((x_i)_{i \in N}\) which converge to 0, but are “cofinally unbounded”, which means that for any \(i \in I\) and any \(C > 0\) there is a \(j \in I, j \geq i\) so that \(|\|x_j\| \geq C|\).

**Hint:** For the second part it might be useful to first prove that for an infinite dimensional Banach \(X\) and functionals \(f_1, f_2, \ldots, f_n \in X^*\) the intersection of the null sets of the \(f_i\)’s contains non zero elements. This can be deduced using basics in linear algebra.

**Proof.** We will first show the hint and use the fact from linear algebra, that for a linear transformation \(T : V \to W\) between two finite dimensional vector spaces it follows that

\[
\dim(V) = \dim(\text{range}(T)) + \dim(\mathcal{N}(T)),
\]

where \(\mathcal{N}(T)\) denotes the null space of \(T\).
Now let $Y$ be any $n+1$ dimensional subspace of $X$ and let $f_i|_Y$ be the restriction of the $f_i$’s to $Y$.

Since $\dim(\text{range}(f_1)) \leq 1$ it follows that $\dim(\mathcal{N}(f_1|_Y)) \geq n$. Now we consider $f_2|_Y \cap \mathcal{N}(f_1)$ and deduce with the same argument
\[ \dim(\mathcal{N}(f_1|_Y) \cap \mathcal{N}(f_2|_Y)) = \dim(\mathcal{N}(f_2|_Y \cap \mathcal{N}(f_1))) \geq \dim(\mathcal{N}(f_1) \cap Y) - 1 \geq n-1. \]
We can continue with this argument and obtain after $n$ steps that
\[ \dim \left( \bigcap_{i=1}^{n} \mathcal{N}(f_i|_Y) \right) \geq 1. \]

Now the construction of our Net. We define as our index set, the typical neighborhood basis os 0 in $X$, i.e.
\[ I = \left\{ \bigcap_{i=1}^{n} \{ x \in X : |f_i(x)| \leq \varepsilon \} : n \in \mathbb{N}, \varepsilon > 0, f_i \in X^*, \|f_i\| = 1, \text{ for } i = 1, \ldots, n \right\}. \]
The order we consider is, as usual, the reversed inclusion.

For each $U = \{ x \in X : |f_i(x)| \leq \varepsilon \} \in I$ we pick an $x_U \in \bigcap_{i=1}^{n} \mathcal{N}(f_i)$ with $\|x_U\| \geq 1/\varepsilon$. Since for every $U \in I$, $x_U \in U$, and $I$ is a neighborhood basis of 0, it follows that $(x_U)_{U \in I}$ is converging to 0, and it is clearly cofinally unbounded (simply make make $\varepsilon$ small enough).

**Problem 6.** Show that if $(x_n)$ converges weakly in a Banach space $X$ to some $x \in X$ then
\[ \lim \inf_{n \to \infty} \|x_n\| \geq \|x\|. \]
Give an example which shows that the inequality can be strict.

**Proof.** Assume that $x_n$ converges weakly to $x$. By the Theorem of Hahn Banach we can find an $x^* \in X^*$ with $\|x^*\| = 1$ and $\langle x^*, x \rangle = \|x\|$. Thus
\[ \lim \inf_{n \to \infty} \|x_n\| \geq \lim \inf_{n \to \infty} \langle x^*, x_n \rangle = \langle x^*, x \rangle = \|x\|. \]

As example take the unit vector basis $(e_n)$ in $c_0$, i.e. as above $e_n = (0, 0, \ldots, 0, 1, 0, \ldots)$ with the “1” being in the $n$ coordinate. By representing $c_0$ as $\ell_1$ (see Problem 2) it follows easily that $(e_n)$ converges weakly to 0, but of course $\|e_n\| = 1$, for all $n \in \mathbb{N}$.

**Problem 7.** Show that $f_n$, with
\[ f_n : [0, 1] \to \mathbb{R}, \quad x \mapsto \sin(2\pi nx), \]
is weakly converging to 0 in $L_p[0, 1], 1 < p < \infty$.

**Proof (Sketch).** Using good old Calculus we observe that for all intervals $I = [a, b] \subset [0, 1]$
\[ \lim_{n \to \infty} \langle f_n, \chi_I \rangle = \lim_{n \to \infty} \int_a^b \sin(2\pi nx) dx = 0. \]
The prove that the set
\[ D = \left\{ A \subset [0, 1] \text{measurable} : \lim_{n \to \infty} \int_A \sin(2\pi nx) dx = 0 \right\} \]
is Dynkin system (standard!). Thus, by Dynkin’s Theorem \(D\) are all measurable subsets of \([0, 1]\) and it follows that
\[
\lim_{n \to \infty} \int g(x) \sin(2\pi nx) \, dx = 0,
\]
for all simple functions \(g\). Now let \(F \in L^*_p[0, 1]\). By the representation theorem for \(L^*_p[0, 1]\) there is an \(f \in L_q[0, 1]\), \(\frac{1}{p} + \frac{1}{q} = 1\), so that
\[
F(h) = \int_0^1 f(x) h(x) \, dx, \quad \text{for all } h \in L_p[0, 1].
\]

Now let \(\varepsilon > 0\) arbitrary. Since the simple functions are dense in \(L_q[0, 1]\), we can choose a simple function \(\tilde{f}\) with \(\|f - \tilde{f}\|_q < \varepsilon\), and thus (note that \(\|f_n\|_p \leq 1\))
\[
\limsup_{n \to \infty} |\langle F, f_n \rangle| = \limsup_{n \to \infty} \left| \int f(x) f_n(x) \, dx \right| \leq \limsup_{n \to \infty} \left[ \left| \int \tilde{f}(x) f_n(x) \, dx \right| + \|f_n\|_p \|f - \tilde{f}\|_q \right] \leq \varepsilon,
\]
which implies the claim since \(\varepsilon > 0\) was arbitrary.