Problems in Topology (Math436)
Prof.: Thomas Schlumprecht Due: 2/28/2008

Read Chapter 5 (beginning).

(1) Problem 5 page 52.

(2) A topological space $(X, \mathcal{T})$ is called *Hausdorff space* or T_2-*space* if for any two points $x, y \in X$, $x \neq y$ there are open sets U and V, with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.
  (a) Show that every metric space is Hausdorff.
  (b) Let X be an infinite set and $\mathcal{T}$ the cofinite topology (i.e. $\mathcal{T} = \{A \subset X : X \setminus A \text{ finite}\} \cup \{\emptyset\}$). Show that $(X, \mathcal{T})$ is not Hausdorff.

(3) Let $(X, \mathcal{T})$ a toplogical space. Using only the definition (this means don't pass to the complement and use results for closures of sets) of A° for $A \subset X$ show for $A, B \subset X$:
  (a) If $A \subset B$ then $A^\circ \subset B^\circ$.
  (b) $(A^\circ)^\circ = A^\circ$.
  (c) $(A \cap B)^\circ = A^\circ \cap B^\circ$.
  (d) $A^\circ \cup B^\circ \subset (A \cup B)^\circ$.

(4) Problem 9 Page 53.

(5) (*) (Definition of toplogies via a *Closure Operator*):
Suppose X is a nonempty set and let $P(X)$ denote the power set of X. The *Kuratovski Closure Operator* is a map

$$C : P(X) \to P(X)$$

satisfying the following conditions
  i) $C(\emptyset) = \emptyset$,
  ii) $A \subset C(A)$, for every $A \in P(X)$,
  iii) $C(C(A)) = C(A)$ for every $A \in P(X)$,
  iv $C(A \cup B) = C(A) \cup C(B)$ for all $A, B \in P(X)$.

Prove:
  (a) C is *isotone* (i.e. if $A \subset B$ then $C(A) \subset C(B)$)
  (b) Define $\mathcal{F} = \{A \subset X : C(A) = A\}$. Show that $\mathcal{T} = \{U : X \setminus U \in \mathcal{F}\}$ is a topology.
  (c) Let $\mathcal{T}$ be the topology in (b). Show that with respect to $\mathcal{T}$ it follows

$$\overline{A} = C(A), \text{ for all } A \in P(X).$$