Problems in Real Analysis II (Math608) **Due: 3/31/10**

Problem 1. (Qualifier Problem) Is there a function $f : [0,1] \to \mathbb{R}$ which is continuous at the rational points but discontinuous at the irrational numbers?

Problem 2. Give a shorter proof of the following former homework problem, using the *Uniform Boundedness Principle.*

Let (x_n) be a weakly convergent sequence in a Banach space X. Then (x_n) is bounded sequence.

Similarly prove, that if (x_n^*) is a w^*-converging sequence in X^*, then (x_n^*) is bounded in X^*.

Problem 3. Recall $\ell_1 = \{x = (x_n) \subset \mathbb{F} : \|x\| = \sum |x_n| < \infty\}$.
Define $X = \{x = (x_n) \in \ell_1 : \sum n|x_n| < \infty\}$.

a) X is dense in ℓ_1.
b) The operator $T : X \to \ell_1$, $(x_n) \mapsto (nx_n)$ is closed but not bounded.
c) $S = T^{-1}$ (T as in (b)) is surjective but not open.
d) Find a new norm on X under which X becomes a Banach space.

Problem 4. The vector space basis of a Banach space is either finite or uncountable.
Remark. In Banach space theory *vector space bases* are mostly not very useful. Banach spaces are not only vector spaces but also have a topology, which can be used for a more appropriate definition of bases:

A sequence (x_n) in a Banach space X is called a *Schauder basis* of X if for every x there is a unique sequence (a_n) in $\mathbb{F}$ so that

$$x = \sum_{n\in\mathbb{N}} a_n x_n.$$

In that case we call (a_n) the *coordinates of x with respect to* (x_n).

Problem 5.

a) Show that every Banach space with a Schauder basis is separable. (It was a long standing open problem whether or not every separable Banach space has a Schauder basis; P. Enflo found separable Banach spaces which do not admit such a basis, nowadays we know that "almost all" separable Banach spaces contain closed subspaces without a Schauder basis)
b) Show that ℓ_p, $1 \le p < \infty$ and c_0 have a Schauder basis (the obvious one)
c) ℓ_∞ does not have a Schauder basis.
(Recall: $\ell_\infty = \{(\xi_j : j\in\mathbb{N}) \subset \mathbb{F} : \|(\xi_j)\|_\infty = \sup_{j\in\mathbb{N}} |\xi_j| < \infty\}$).

Problem 6. Prove that for every separable Banach space there is a surjection

$$S : \ell_1 \to X.$$

Show that every separable Banach space is isomorphic to some quotient of ℓ_1.

Problem 7. Let $\mathbb{F} = \mathbb{R}$ and define for $n \in \mathbb{N}$

$$E_n := \big\{ f \in C[0,1] : \exists x_0 \in [0,1] \quad |f(x) - f(x_0)| \leq n|x - x_0|, \text{ for all } x \in [0,1] \big\}.$$

a) E_n is nowhere dense and closed.
b) The set of all $f \in C[0,1]$ which are nowhere differentiable is residual in $C[0,1]$.