Problems in Real Variables, II (Math608), Solutions
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Problem 1. (Qualifier Problem) Is there a function $f : [0,1] \to \mathbb{R}$ which is continuous at the rational points but discontinuous at the irrational numbers?
Proof. Answer no. Assume that such an $f : [0,1] \to \mathbb{R}$ existed.
For $x \in [0,1]$ note that

$$\begin{aligned} &f \text{ discontinuous in } x \\ &\iff \exists \varepsilon > 0 \, \forall \delta > 0 \, \exists y, z \in [x-\delta, x+\delta] \cap [0,1] \quad |f(y) - f(z)| \geq \varepsilon \\ &\iff \exists k \in \mathbb{N} \, \forall \delta > 0 \, \exists y, z \in [x-\delta, x+\delta] \cap [0,1] \quad |f(y) - f(z)| \geq 1/k \\ &\iff x \in \bigcup_{k\in\mathbb{N}} A_k \text{ with} \end{aligned}$$

$$A_k = \{x \in [0,1] : \forall \delta > 0 \exists \, y, z \in [x-\delta, x+\delta] \cap [0,1] \quad |f(y) - f(z)| \geq 1/k\}.$$

Now we claim that A_k is closed for every $k \in \mathbb{N}$. Indeed, if $(x_j) \subset A_k$, so that $x = \lim_{j\to\infty} x_j \in [0,1]$ exists, and if $\delta > 0$ arbitrary, then there exists a $j_0 \in \mathbb{N}$ so that $|x - x_{j_0}| < \delta/2$ and

$$y, z \in [x_{j_0} - \delta/2, x_{j_0} + \delta/2] \cap [0,1] \subset [x-\delta, x+\delta] \cap [0,1],$$

so that $|f(y) - f(z)| \geq 1/k$.

For $n \in \mathbb{N}$ let B_n be the singleton $\{q_n\}$, where $\{q_j : j \in \mathbb{N}\}$ is an enumeration of all the rational numbers in $[0,1]$. We can write $[0,1] = \bigcup_{k\in\mathbb{N}} A_k \cup \bigcup_{n\in\mathbb{N}} B_n$. By the Baire Category theorem some of the A_n's or some of the B_n must have an non empty open interior, and thus contain a non degenerated interval. But this is obviously false.

Problem 2. Give a shorter proof of the following former homework problem, using the *Uniform Boundedness Principle.*

Let (x_n) be a weakly convergent sequence in a Banach space X. Then (x_n) is bounded sequence.

Similarly prove, that if (x_n^*) is a w^*-converging sequence in X^*, then (x_n^*) is bounded in X^*.
Proof. For $n \in \mathbb{N}$ consider x_n as an operator on X^* into $\mathbb{F}$ via the canonical map of X into X^{**}, i.e. $\kappa(x_n) : X^* \to \mathbb{F}$. Since for every $x^* \in X^*$, the sequence $\big(\kappa(x_n)(x^*)\big) = \big(x^*(x_n)\big)$ is converging and thus bounded, it follows that $\big(\kappa(x_n)\big)$ is bounded in X^{**}, and thus, by the Hahn Banach Theorem, (x_n) is bounded in X.

A w^* converging sequence (x_n^*) is a sequence of operators from X to $\mathbb{F}$, and, thus, the same argumentation as before will work.

Problem 3. Recall $\ell_1 = \{x = (x_n) \subset \mathbb{F} : \|x\| = \sum |x_n| < \infty\}$.
Define $X = \{x = (x_n) \in \ell_1 : \sum n|x_n| < \infty\}$.

a) X is dense in ℓ_1.

b) The operator $T : X \to \ell_1$, $(x_n) \mapsto (nx_n)$ is closed but not bounded.
c) $S = T^{-1}$ (T as in (b)) is surjective but not open.
d) Find a new norm on X under which X becomes a Banach space.

Proof. (a) note that for any $x = (x_i) \in \ell_1$ and any $k \in \mathbb{N}$

$$x^{(k)} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in X$$

and $\|x^{(k)} - x\|_{\ell_1} \to 0$ if $k \nearrow \infty$.
(b) T is not bounded, because $\frac{1}{n} e_n \to 0$ but $T(\frac{1}{n} e_n) = e_n \not\to 0$ (e_n has coordinate 1 in n-place and is 0 elsewhere).

Assume $x^{(n)} \to x$ and $T(x^{(n)}) \to y$ for $n \nearrow \infty$. To show $T(x) = y$. Since

$$\|Tx^{(n)} - y\| = \sum_{k \in \mathbb{N}} |kx_k^{(n)} - y_k| = \sum_{k \in \mathbb{N}} k \Big| x_k^{(n)} - \frac{y_k}{k} \Big|$$

it follows (after letting $n \to \infty$) that $(x_k) = (y_k/k)$, for each $k \in \mathbb{N}$, and, thus, $T(x) = y$.
(c) The inverse of T is

$$S : \ell_1 \to X, \text{ with } (x_k) \mapsto (x_k/k).$$

Let B be the open unit ball around 0. Then the vectors $x^{(i)} = 2e_i/i$ converge in norm to 0 but $x_i \notin S(B)$ therefore $S(B)$ cannot be a neighborhood of 0.
(d) For $x = (x_n) \in X$ define $|||x||| = \sum_{n \in \mathbb{N}} n|x_n|$. Then X can be seen as $X = L_1(\mu)$ where μ is the measure on N (with the power set), defined by $\mu(A) = \sum_{j \in A} j$, for $A \subset N$, in other words μ is the measure whose Radon Nikodym derivative with respect to the counting measure is the function $f : \mathbb{N} \to \mathbb{N}$, $k \mapsto k$.

Problem 4. The vectorspace basis of a Banach space is either finite or uncountable.
Proof. Assume X is a Banach space which has an infinite but countable basis, say $(x_n : n \in \mathbb{N})$. For $n \in \mathbb{N}$ let X_n be the (finite dimensional) subspace of X spanned by the first n basis vectors, i.e.

$$X_n = \Big\{ \sum_{j=1}^{n} a_j x_j : a_j \in \mathbb{F}, \text{ for } j = 1, 2, \ldots n \Big\}.$$

By former homework problem X_n is closed (since it is finite dimensional). On the other hand, since each $x \in X$ is a **finite** linear combination of the x_n's we conclude that $X = \bigcup X_n$. By the Baire Cathegory Theorem there exists an n so that X_n contains an open ball, say

$$B_\varepsilon(x) \subset X_n, \text{ with } \varepsilon > 0 \text{ and } x = \sum_{j=1}^{n} a_j x_j \in X_n.$$

But note that $y = x + \frac{\varepsilon}{2} \frac{x_{n+1}}{\|x_{n+1}\|} \in B_\varepsilon(x)$ but $y \notin X_n$ (uniqueness of writing y as linear combination of x_j's).

Problem 5.

a) Show that every Banach space with a Schauder basis is separable. (It was a long standing open problem whether or not every separable Banach space has a Schauder basis; P. Enflo found separable Banach spaces which do not admit such a basis, nowadays we know that "almost all" separable Banach spaces contain closed subspaces without a Schauder basis)

b) Show that ℓ_p, $1 \le p < \infty$ and c_0 have a Schauder basis (the obvious one).

c) ℓ_∞ does not have a Schauder basis.

Proof. (a) Let (x_n) be a basis of X. We first observe that

$$\tilde{X} = \Big\{ \sum_{j=1}^{n} a_j x_j : n \in \mathbb{N},\, a_j \in \mathbb{F}, \text{ for } j = 1, 2, \dots n \Big\}$$

is dense in X (any $x \in X$, can be written as converging series $\sum_{j=1}^{\infty} a_j x_j$).

Secondly define and let $D \subset \mathbb{F}$ be dense (for example $D = \mathbb{Q}$, if $\mathbb{F} = \mathbb{R}$, and $\mathbb{F} = \mathbb{Q} + i\mathbb{Q}$, if $\mathbb{F} = \mathbb{C}$). and note that

$$\widehat{X} = \Big\{ \sum_{j=1}^{n} a_j x_j : n \in \mathbb{N},\, a_j \in D, \text{ for } j = 1, 2, \dots n \Big\}$$

is dense in $\tilde{X}$. Indeed, if $x = \sum_{j=1}^{n} a_j x_j \in \tilde{X}$, chooses sequence $(d_k^{(j)} : k \in \mathbb{N})$, for $j = 1, 2, \dots, n$ with $a_j = \lim_{k\to\infty} d_k^{(j)}$, for $j = 1, 2. \dots, n$. Then

$$\Big\| x - \sum_{j=1}^{n} d_k^{(j)} x_j \Big\| \le \sum_{j=1}^{n} |a_j - d_k^{(j)}| \|x_j\| \to_{k\to\infty} 0.$$

(b) We show that the unit vector basis (e_j) is a Schauder basis for ℓ_p, $1 \le p < \infty$, a similar argument works for c_0.

Let $x = (\xi_j : j \in \mathbb{N}) \in \ell_p$, then

$$\Big\| x - \sum_{j=1}^{n} \xi_j e_j \Big\| = \Big(\sum_{j=n+1}^{\infty} |\xi|^p \Big)^{1/p} \to_{n\to\infty} 0.,$$

and thus

$$x = \sum_{j=1}^{\infty} \xi_j e_j.$$

The uniqueness of that representation is clear since $y = (\eta_j) \ne x$ if and only if $\eta_j \ne \xi_j$ for some $j \in \mathbb{N}$.

(c) We will show that ℓ_∞ is not separable. Then our claim follows from part (a).

Consider the uncountable set $\{-1, 1\}^{\mathbb{N}}$ of all ±1-valued sequences, for $x, y \in \{-1, 1\}^{\mathbb{N}}$, with $x \ne y$, we have $\|x - y\|_\infty = 2$. Let $D \subset \ell_\infty$ be dense. Since for any $x \in \{-1, 1\}$ there is a $d_x \in D$ so that $\|x - d_x\| < 1$, D must be uncountable.

Problem 6. Prove that for every separable Banach space there is a surjection

$$S : \ell_1 \to \; X.$$

Show that X is isomorphic to a quotient space of ℓ_1.

Proof. Let (x_n) dense sequence in the sphere of X (or in B_X), and define

$$S : \ell_1 \to X, \qquad z = (\eta_n) \mapsto \sum_{n\in\mathbb{N}} \eta_n x_n.$$

S is well define since above sums are absolutely converging and since for $z = (\eta_n) \in \ell_1$.

$$\|S(z)\| = \Big\| \sum_{n=1}^{\infty} \eta_n x_n \Big\| \leq \sum_{n=1}^{\infty} |\eta_n| = \|z\|,$$

it follows that S is bounded and $\|S\| \leq 1$.

In order to show surjectivity we let $x \in X$, and by induction we choose numbers r_j and $n_j \in \mathbb{N}$, for $j \in \mathbb{N} \cup \{0\}$, so that

$$r_0 \leq \|x\|,\ 0 \leq r_j < 2^{-j}, \text{ and } \Big\| x - \sum_{j=0}^{n} r_j x_{n_j} \Big\| \leq 2^{-n-1}, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

It follows that the series $\sum_{j=0}^{n} r_j x_j$ converges to x. We can assume that $n_j \to= n_i$ is $i \neq j$, and we define Define $z = (\zeta_n) \in \ell_1$, by $\zeta_n = r_{n_j}$, if $n = n_j$ for some $j \in \mathbb{N}$, and otherwise $\zeta_n = 0$. Then $S(z) = x$.

The open mapping theorem yields that there is an $\varepsilon > 0$ so that $\varepsilon B_X \subset S(B_{\ell_1})$. Let

$$Y = \mathcal{N}_S = \{z \in \ell_1 : S(z) =\}.$$

We define

$$T : X \to \ell_1/Y, T(x) := S^{-1}(\{x\}).$$

Note that the set $S^{-1}(\{x\})$ can be written as

$$S^{-1}(\{x\} = z + Y, \text{ where } z \in \ell_1 \text{ is such that } S(z) = x.$$

T is bijective and for any $x \in B_X$ there exists a $z \in \ell_1$, so that $\|z\| \leq 1/\varepsilon$ and $S(z) = x$, and thus

$$\|T(x)\| = \inf_{y\in Y} \|z - y\| \leq 1/vp.$$

It follows that T is bounded, and since it is also bijective it follows from the Closed Graph Theorem that T is an isomorphism.

Problem 7. Let $\mathbb{F} = \mathbb{R}$ and define for $n \in \mathbb{N}$

$$E_n := \big\{ f \in C[0,1] : \exists x_0 \in [0,1] \quad |f(x) - f(x_0)| \leq n|x - x_0|, \text{ for all } x \in [0,1] \big\}.$$

a) E_n is nowhere dense in $C[0,1]$.
b) The set of all $f \in C[0,1]$ which are nowhere differentiable is residual in $C[0,1]$.

Proof. a) We first observe that E_k is closed for any $k \in \mathbb{N}$. Indeed, assume $(f_n) \subset E_k$ converges to some $f \in C[0,1]$, and choose $x_0^{(n)} \in [0,1]$ so that

$$|f_n(x) - f_n(x_0^{(n)})| \le k|x - x_0^{(n)}|, \text{ for all } x \in [0,1].$$

Since $[0,1]$ is compact we can assume without loss of generality that $x_0^{(n)}$ converges to some $x_0 \in [0,1]$. It follows therefore for all $x \in [0,1]$

$$|f(x) - f(x_0)| = \lim_{n\to\infty} |f_n(x) - f_n(x_0^{(n)})| \le k \liminf_{n\to\infty} |x - x_0^{(n)}| = k|x - x_0|,$$

which proves that $f \in E_k$.

In order to show that E_k is nowhere dense, we need show that for a given $f \in E_k$ and $\varepsilon > 0$, there is a function $\tilde{f} \in C[0,1]$, with $\tilde{f} \notin E_k$, and $\|f - \tilde{f}\| \le \varepsilon$. Let $\text{zigzag}_{(\varepsilon,k)}$ be a function in $C[0,1]$ which is piecewise linear, has either derivaitve $\pm k$ and $\|\text{zigzag}_{(\varepsilon,k)}\|_\infty = \varepsilon$. For example

$$\text{zigzag}_{(\varepsilon,k)}(x) = kx\chi_{[0,\varepsilon/k]} + (2\varepsilon - kx)\chi_{(\varepsilon/k,3\varepsilon/k]} + (kx - 4\varepsilon)\chi_{(3\varepsilon/k,5\varepsilon/k]} \cdots$$

Then proceed as follows: Let $f \in E_n$ fined (using Stone Weierstrass) a polynomial p so that $||f - p|| < \varepsilon/2$. Define $C = \sup_{x\in[0,1]} |p(x)|$ and then choose $N \in \mathbb{N}$, so that $N > C + k$. It follows

$$\tilde{f} := p + \text{zigzag}_{(\varepsilon/2/N)} \notin E_k.$$

Indeed for each $x \in [0,1]$ there is a y so that $|\text{zigzag}_{(\varepsilon/2/N)}(x) - \text{zigzag}_{(\varepsilon/2/N)}(y)| \ge N|x - y|$, and thus

$$|\tilde{f}(x) - \tilde{f}(y)| \ge |\text{zigzag}_{(\varepsilon/2/N)}(x) - \text{zigzag}_{(\varepsilon/2/N)}(y)| - |p(x) - p(y)| \ge N||x-y| - C|x-y| > k|x-y|.$$

b) follows from the Baire Category theorem and the easily proven fact that $\bigcup_k E_k$ contains the functions which are differentiable in at least one point.