**Problems to Introduction to Real Analysis, (Math446)**

Due: 12/2/04

$(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are topological spaces.

**Problem 1.** Let $\mathcal{B}$ be a subbasis of $\mathcal{S}$ and $f : X \to Y$ a map. Then the following are equivalent.

a) $f$ is $(\mathcal{T}, \mathcal{S})$-continuous,
b) $f^{-1}(C)$ is closed whenever $C \subset Y$ is $\mathcal{S}$-closed,
c) $f^{-1}(U)$ is in $\mathcal{T}$ for all $U \in \mathcal{B}$.

**Problem 2.** Let $\mathcal{T} \otimes \mathcal{S}$ denote the product topology on $X \times Y$ for a sequence $(x_n, y_n)$ and $(x, y) \in X \times Y$ the following are equivalent.

a) $(x_n, y_n)$ converges to $(x, y)$ in $\mathcal{T} \otimes \mathcal{S}$,
b) $x_n$ converges to $x$ in $\mathcal{T}$ and $(y_n)$ converges to $y$ in $\mathcal{S}$,

**Problem 3.** If $f : X \to Y$ is continuous, and $K \subset X$ is compact then $f(K)$ is compact.

**Problem 4.** Assume that $(X, \mathcal{T})$ is *Hausdorff*, meaning that for all $x, z \in X$, $x \neq z$ then there are open sets $U$ and $V$, so that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. (note that metric spaces are Hausdorff).

Now let $K \subset X$ compact and $x \in X \setminus K$, prove that there are open sets $U$ and $V$ so that $x \in U$, $K \subset V$ and $U \cap V = \emptyset$.

Deduce that in a Hausdorff space every compact set must be closed.

**Problem 5.** Assume that $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are compact and Hausdorff and $f : X \to Y$ is continuous and bijective.

Show that $f$ is actually a homeomorphism.

**Problem 6.** $(*)$ Consider on $\{0, 1\}$ the discrete topology. And let

$$X = \{0, 1\}^{[0,1]} = \{f : [0, 1] \to \{0, 1\}\}$$

be endowed with the product topology.

For a sequence $(f_n) \subset X$ and $f \in X$ show that

$$f_n \to f \iff \forall A \subset [0, 1] \text{ finite } \exists n \in \mathbb{N} \, \forall m \geq n \, \forall a \in A \quad f_m(a) = f(a).$$

Define

$$A = \big\{f : [0, 1] \to \{0, 1\} : \#\{a \in [0, 1] : f(a) = 1\} \text{ countable}\big\}.$$

Show that every sequence in $A$ has a convergent subsequence whose limit is in $A$, but that $A$ is not compact.