Problems to Introduction to Real Analysis, (Math447)

Due:4/21/2005

Problems 24 -28 on Sheet

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be measurable, and integrable on every bonded measurable set with respect to Lebesgue measure (meaning that $\int_A |f|dx < \infty$ for any bounded measurable set $A \subset \mathbb{R}$.

Assume that for any open bounded set $U \subset \mathbb{R}$ it follows that

$$(*) \qquad -1 \le \int_U f(x)dx \le 1.$$

a) show that (*) holds for any bounded measurable set.

b) Show that $f \in L_1(\mathbb{R})$ and that $\|f\|_{L_1} \le 1$.

Problem 7. If $f \in L_1(0, \infty)$ (with Lebesgue measure), define

$$g(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad 0 < s < \infty.$$

Prove that $g(s)$ is differentiable on $(0, \infty)$, and has derivative

$$-\int_0^\infty e^{-st} t f(t)\, dt.$$

(The inequality $1 - e^{-x} \le x$, $x \in [0, \infty)$ may be used without proof, if needed)

Problem 8. Let $f, g \in L_1(\mu)$, where $(X, \mathcal{M}, \mu)$ is a measure space, assume that $\int_\Omega f(\omega)\mathrm{d}\mu(\omega) < \int_\Omega g(\omega)\mathrm{d}\mu(\omega)$. Show that there is a measurable $E \subset \Omega$ of strictly positive measure so that

$$\sup_{x \in E} f(x) < \inf_{x \in E} g(x)\,.$$