Problems in Real Analysis II (Math608) **Due: 4/7/10**

Problem 1. There is a meager subset of $\mathbb{R}$ whose complement has Lebesgue's measure zero.

Problem 2. Assume that $\|\cdot\|$ and $|||\cdot|||$ are two norms on a vector space X so that $\|\cdot\| \le |||\cdot|||$ an so that under both norms X is complete. Show that the norms are equivalent, i.e. that there is a constant $c \ge 1$ so that $|||\cdot||| \le c\|\cdot\|$.

Problem 3. Let X and Y be Banach spaces. If $T : X \to Y$ is a linear map such that $f \circ T \in X^*$ for all $f \in Y^*$. Show that T is bounded.

Problem 4. Let (x_n) be a Schauder basis of a Banach space X (see Homework 7). For $x \in X$ and $n \in \mathbb{N}$ define

$$P_n(x) = \sum_{j=1}^{n} a_j x_j, \text{ where } x \text{ has (unique) expansion } x = \sum_{i=1}^{\infty} a_i x_i.$$

a) Prove that P_n is a linear bounded map, and that $M := \sup \|P_n\| < \infty$.
Hint: consider the norm

$$|||x||| = \sup_{n\in\mathbb{N}} \Big\| \sum_{j=1}^{n} a_i x_i \Big\|, \text{ where } x \text{ has (unique) expansion } x = \sum_{i=1}^{\infty} a_i x_i.$$

b) Prove that for $n \in \mathbb{N}$ the *n-th coordinal functional*

$$x_n^* : X \to \mathbb{F},\ x \mapsto a_n, \text{ where } x \text{ has (unique) expansion } x = \sum_{i=1}^{\infty} a_i x_i,$$

is in X^* and if $\inf_{n\in\mathbb{N}} \|x_n\| > 0$ then $\sup_{n\in\mathbb{N}} \|x_n^*\| < \infty$.

Problem 5. Let X be a non empty set. We call a set $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$ a *filter on* X if for all $A, B \in \mathcal{F}$ there is a $C \in \mathcal{F}$ so that $C \subset A \cap B$. Note that in a topological space X a neighborhood basis of some point $x \in X$ is a filter. We call a filter $\mathcal{F}$ and *ultrafilter* if it is maximal, i.e. if for any $A \in \mathcal{P}(X) \setminus \mathcal{F}$ $\mathcal{F} \cup \{A\}$ is not anymore a filter.

a) Show that every filter $\mathcal{F}$ can be extended to an ultra filter.
b) Let $\mathcal{F}$ be a filter. Then

$$\mathcal{F} \text{ is an ultrafilter} \iff \forall A \in \mathcal{P}(X) \quad A \in \mathcal{F} \text{ or } A^c \in \mathcal{F}.$$

c) If X is infinite there are *nontrivial ultrafilter* $\mathcal{U}$, i.e. with the property that $\mathcal{U}$ does not contain finite set (or equivalently, (why?) singletons).
d) Let $x \in \ell_\infty$ and let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. Then there exists an $r = r(\mathcal{U}, x) \in \mathbb{R}$, so that for all $\varepsilon > 0$ there is an $N \in \mathcal{U}$ so that $|x_n - r| < \varepsilon$ for all $n \in N$.

e) Think of an ultrafilter to be a directed set (reversed inclusion) and pick for every $N \in \mathcal{U}$ and element $k_N \in N$. Then for all $x = (x_n) \in \ell_\infty$

$$r(\mathcal{U}, x) = \lim_{U \in \mathcal{U}} x_{k_U}.$$

f) Show that for every ultrafilter the map

$$\mathcal{U}(\cdot) : X \to \mathcal{F}, \quad x \to r(\mathcal{U}, x)$$

is bounded and linear (and thus an element of ℓ_∞^*).

Problem 6. Let X be locally convex space over $\mathbb{R}$, $A \subset X$ closed and convex and $K \subset X$ compact and convex, and assume that A and K are disjoint and both non empty. Show that there is an $f \in X^*$ so that

$$\sup_{x \in A} f(x) < \inf_{x \in K} f(x).$$

Give an example which shows that one cannot replace K compct by only K closed (of course all other conditions are satisfied).

Problem 7. 45/page 170.

Problem 8. 49/page 170.