**Problems in Real Analysis II (Math608)**

**Due: 4/7/10**

**Problem 1.** There is a meager subset of \( \mathbb{R} \) whose complement has Lebesgue’s measure zero.

**Problem 2.** Assume that \( \| \cdot \| \) and \( ||| \cdot ||| \) are two norms on a vector space \( X \) so that \( \| \cdot \| \leq ||| \cdot ||| \) an so that under both norms \( X \) is complete. Show that the norms are equivalent, i.e. that there is a constant \( c \geq 1 \) so that \( ||| \cdot ||| \leq c \| \cdot \| \).

**Problem 3.** Let \( X \) and \( Y \) be Banach spaces. If \( T : X \to Y \) is a linear map such that \( f \circ T \in X^\ast \) for all \( f \in Y^\ast \). Show that \( T \) is bounded.

**Problem 4.** Let \( (x_n) \) be a Schauder basis of a Banach space \( X \) (see Homework 7). For \( x \in X \) and \( n \in \mathbb{N} \) define

\[
P_n(x) = \sum_{j=1}^{n} a_j x_j, \text{ where } x \text{ has (unique) expansion } x = \sum_{i=1}^{\infty} a_i x_i.
\]

a) Prove that \( P_n \) is a linear bounded map, and that \( M := \sup_n \|P_n\| < \infty \).

**Hint:** consider the norm

\[
\|x\| = \sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} a_j x_j \right\|, \text{ where } x \text{ has (unique) expansion } x = \sum_{i=1}^{\infty} a_i x_i.
\]

b) Prove that for \( n \in \mathbb{N} \) the \( n \)-th coordinal functional

\[
x_n^* : X \to \mathbb{F}, \ x \mapsto a_n, \text{ where } x \text{ has (unique) expansion } x = \sum_{i=1}^{\infty} a_i x_i,
\]

is in \( X^\ast \) and if \( \inf_{n \in \mathbb{N}} \|x_n\| > 0 \) then \( \sup_{n \in \mathbb{N}} \|x_n^*\| < \infty \).

**Problem 5.** Let \( X \) be a non empty set. We call a set \( \mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\} \) a filter on \( X \) if for all \( A, B \in \mathcal{F} \) there is a \( C \in \mathcal{F} \) so that \( C \subset A \cap B \). Note that in a topological space \( X \) a neighborhood basis of some point \( x \in X \) is a filter. We call a filter \( \mathcal{F} \) and ultrafilter if it is maximal, i.e. if for any \( A \in \mathcal{P}(X) \setminus \mathcal{F} \mathcal{F} \cup \{A\} \) is not anymore a filter.

a) Show that every filter \( \mathcal{F} \) can be extended to an ultra filter.

b) Let \( \mathcal{F} \) be a filter. Then

(\( \mathcal{F} \) is an ultrafilter \( \iff \forall A \in \mathcal{P}(X) \quad A \in \mathcal{F} \) or \( A^c \in \mathcal{F} \).

**c)** If \( X \) is infinite there are nontrivial ultrafilter \( \mathcal{U} \), i.e. with the property that \( \mathcal{U} \) does not contain finite set (or equivalently, (why?) singletons).

**d)** Let \( x \in \ell_\infty \) and let \( \mathcal{U} \) be an ultrafilter on \( \mathbb{N} \). Then there exists an \( r = r(\mathcal{U}, x) \in \mathbb{R} \), so that for all \( \varepsilon > 0 \) there is an \( N \in \mathcal{U} \) so that \( |x_n - r| < \varepsilon \) for all \( n \in N \).
e) Think of an ultrafilter to be a directed set (reversed inclusion) and pick for every $N \in \mathcal{U}$ and element $k_N \in N$. Then for all $x = (x_n) \in \ell_\infty$,

$$r(\mathcal{U}, x) = \lim_{U \in \mathcal{U}} x_{k_U}.$$  

f) Show that for every ultrafilter the map

$$\mathcal{U}(\cdot) : X \to \mathcal{F}, \quad x \to r(\mathcal{U}, x)$$

is bounded and linear (and thus an element of $\ell_\infty^*.$)

**Problem 6.** Let $X$ be locally convex space over $\mathbb{R}$, $A \subset X$ closed and convex and $K \subset X$ compact and convex, and assume that $A$ and $K$ are disjoint and both non empty. Show that there is an $f \in X^*$ so that

$$\sup_{x \in A} f(x) < \inf_{x \in K} f(x).$$

Give an example which shows that one cannot replace $K$ compact by only $K$ closed (of course all other conditions are satisfied).

**Problem 7.** 45/page 170.

**Problem 8.** 49/page 170.