Problems in Real Variables, II (Math608), Solutions
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Problem 1. There is a meager subset of $\mathbb{R}$ whose complement has Lebesgue's measure zero.
Proof. Given any $\varepsilon > 0$ we will construct a "fat cantor set" $S \subset [0,1]$, which is closed, nowhere dense and has measure $m(S) \geq 1-\varepsilon$.

Having done this we are able to choose for any $k \in \mathbb{N}$ and any $m \in \mathbb{Z}$ a closed, nowhere dense set $S_{(k,m)} \subset [m, m+1]$ with $m(S_{(k,m)}) \geq 1 - \frac{1}{k}$. Therefore $S = \bigcup_{k\in\mathbb{N}, m\in\mathbb{Z}} S_{(k,m)}$ is meager but has full measure.

In order to construct S for given ε we let ε_i be a sequence of positive numbers with $\sum_{i\in N} 2^i \varepsilon_i = \varepsilon$. We let $S_0 = [0,1]$ and assuming we defined S_k being the union of 2^k closed intervals, say $[a_i, b_i]$, $i = 1, \ldots 2^k$, with $b_i - a_i < 2^{-k}$, we take out of each $[a_i, b_i]$ the middle interval of length $\min(\varepsilon_i, (b_i - a_i)/3)$. S_{k+1} is then the union of 2^{k+1} intervals, of lengths smaller than 2^{-k-1}.

We notice for $S = \bigcap S_k$ that S is closed, does not contain an interval, hence it is nowhere dense, and that

$$m(S) = \lim_{k\to\infty} m(S_k) \geq \lim_{k\to\infty} 1 - \sum_{i=1}^{k} 2^i \varepsilon_i = 1 - \varepsilon.$$

Note also that $[0,1] \setminus S$ is dense in $[0,1]$ and thus $S^0 = \emptyset$.

Problem 2. Assume that $\|\cdot\|$ and $|||\cdot|||$ are two norms on a vector space X so that $\|\cdot\| \leq |||\cdot|||$ an so that under both norms X is complete. Show that the norms are equivalent, i.e. that there is a constant $c \geq 1$ so that $|||\cdot||| \leq c\|\cdot\|$.
Proof. Consider the identity

$$I : (X, |||\cdot|||) \to (X, ||\cdot||), \quad x \mapsto x.$$

I is bounded and thus the graph is closed. Now the inverse map

$$I^{-1}(X, ||\cdot||) \to (X, |||\cdot|||), \quad x \mapsto x$$

has, up to reflection, the same graph as I and since it is closed, it follows from the closed graph theorem that also I^{-1} is bounded. But this simply means that there is a $c \geq 1$ so that $|||\cdot||| \leq c\|\cdot\|$.

Problem 3. Let X and Y be Banach spaces. If $T : X \to Y$ is a linear map such that $f \circ T \in X^*$ for all $f \in Y^*$. Show that T is bounded.
Proof. For $x \in X$ consider the linear bounded map $T_x : Y^* \to \mathbb{F}$, $f \mapsto f \circ T(x)$. Apply the Uniform Boundedness Principle to the family

$$\{T_x : x \in B_X\} \subset L(Y^*, \mathbb{F})$$

and deduce that

$$\begin{aligned}\infty > \sup_{x\in B_X} ||T_x|| &= \sup_{x\in B_x}\sup_{f\in B_{Y^*}} |T_x(f)|\\ &= \sup_{f\in B_{Y^*}}\sup_{x\in B_x} f(T(x))\\ &= \sup_{x\in B_x} \|T(x)\| \text{ (By Hahn-Banach)}\\ &= \|T\|.\end{aligned}$$

Problem 4. Let (x_n) be a Schauder basis of a Banach space X (see Homework 7). For $x \in X$ and $n \in \mathbb{N}$ define

$$P_n(x) = \sum_{j=1}^n a_j x_j, \text{ where } x \text{ has (unique) expansion } x = \sum_{i=1}^\infty a_i x_i.$$

a) Prove that P_n is a linear bounded map, and that $M := \sup \|P_n\| < \infty$.
Hint: consider the norm

$$|||x||| = \sup_{n\in\mathbb{N}} \Big\| \sum_{j=1}^n a_i x_i \Big\|, \text{ where } x \text{ has (unique) expansion } x = \sum_{i=1}^\infty a_i x_i.$$

b) Prove that for $n \in \mathbb{N}$ the n-*th coordinal functional*

$$x_n^* : X \to \mathbb{F},\, x \mapsto a_n, \text{ where } x \text{ has (unique) expansion } x = \sum_{i=1}^\infty a_i x_i,$$

is in X^* and if $\inf_{n\in\mathbb{N}} \|x_n\| > 0$ then $\sup_{n\in\mathbb{N}} \|x_n^*\| < \infty$.

Proof: (a) For $x \in X$,

$$|||x||| = \sup_{m\in\mathbb{N}} \|P_m(x)\| = \sup_{m\in\mathbb{N}} \Big\| \sum_{j=1}^n a_j x_j \Big\|,$$

where $x = \sum_{j=1}^\infty a_j x_j$ is the unique representation of x as infinite linear combination of the x_j's. It is easy to show (but has to be shown) that $|||\cdot|||$ is a norm on X and that $\|x\| \le |||x|||$, for all $x \in X$. Moreover the maps

$$P_m : (X, |||\cdot|||) \to X, \qquad (X, |||\cdot|||), \sum_{j=1}^\infty a_j x_j \mapsto \sum_{j=1}^m a_j x_j,$$

are bounded and

$$|||P_m||| = \sup_{|||x|||\le 1} |||P_m(x)||| = 1.$$

Our main problem is to show that $(X, |||\cdot|||)$ is complete. Once we have done this we deduce from the Closed Graph Theorem that the Identity

$$I : (X, \|\cdot\|) \to (X, |||\cdot|||)$$

is bounded which implies that for some $C \geq 1$

$$|||x||| = \sup_{m\in\mathbb{N}} \|P_m(x)\| \leq C\|x\|, \text{ for all } x \in X.$$

In order to show completeness of $(X, |||\cdot|||)$ let $y^{(n)}$ be a $|||\cdot|||$ Cauchy sequence in $(X, |||\cdot|||)$. Since P_m is bounded on $(X, |||\cdot|||)$, for fixed m $(P_m(y^{(n)})$ is a Cauchy sequence in $(X, |||\cdot|||)$ and thus (image of P_m is finite dimensional) convergent to some $z_m = \sum_{j=1} \beta_{(m,j)} x_j$. Since $P_m \circ P_n = P_{\min(m,n)}$, it follows that $\beta_{(m,j)}$ depends only of j and thus we can write $z_m = \sum_{j=1} \beta_j x_j$. We now claim that z_m converges in X. Indeed, let $\varepsilon > 0$ be arbitrary. Choose $k_0 \in \mathbb{N}$ so that $|||y^{(s)} - y^{(t)}||| < \varepsilon/3$ for all $s, t \geq k_0$ and then choose $m_0 \in \mathbb{N}$ so that

$$\|(P_M - P_m)(y^{(n_0)})\| = \|\sum_{j=m+1}^{M} b_j x_j\| < \varepsilon/3 \text{ for all } M > m \geq m_0.$$

Then we conclude that for all $M > m \geq m_0$.

$$\|\sum_{m+1}^{M} b_j x_j\| = \lim_{k\to\infty} \|(P_M - P_m)y^{(k)}\| \leq \|(P_M - P_m)y^{(k_0)}\| + \limsup_{k\to\infty} |||y^{(k_0)} - y^{(k_0)}||| < \varepsilon.$$

Thus we conclude that $z = \lim_{m\to\infty} z_m = \sum_{j=1}^{\infty} b_j x_j$ converges. We also note that for $n \in \mathbb{N}$

$$|||z - y^{(n)}||| = \sup_{m\in\mathbb{N}} \|P_m(z - y^{(n)})\| = \sup_{m\in\mathbb{N}} \limsup_{N\to\infty} \|P_m(y^{(N)} - y^{(n)})\| \leq \sup_{n\leq N} |||P_m(y^{(N)} - y^{(n)})||| \to_{n\to\infty} 0,$$

which shows that $y^{(n)}$ converges to z and finishes the proof of our claim.
(b) Note that $(P_0 = 0)$ for $m \in \mathbb{N}$

$$P_m(x) - P_{m-1}(x) = x^*_m(x) x_m,$$

and if we choose $f^*_m \in X^*$ with $f^*_m(x_m) = 1$ and $||f^*_m|| = 1/\|x_m\|$, it follows that

$$x^*_m(x) = f_m*(P_m(x) - P_{m-1}(x)), \text{ and thus } x^*_m \in X^* \text{ with } \|x^*_m\} \leq 2 \sup_{k\in\mathbb{N}} \frac{\|P_k\|}{\|x_k\|},$$

which implies the claim.

Problem 5. Let X be a non empty set. We call a set $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$ a *filter on* X if for all $A, B \in \mathcal{F}$ there is a $C \in \mathcal{F}$ so that $C \subset A \cap B$. Note that in a topological space X a neighborhood basis of some point $x \in X$ is a filter. We call a filter $\mathcal{F}$ and *ultrafilter* if it is maximal, i.e. if for any $A \in \mathcal{P}(X) \setminus \mathcal{F}$ $\mathcal{F} \cup \{A\}$ is not anymore a filter.

a) Show that every filter $\mathcal{F}$ can be extended to an ultra filter.
b) Let $\mathcal{F}$ be a filter. Then

$$\mathcal{F} \text{ is an ultrafilter} \iff \forall A \in \mathcal{P}(X) \quad A \in \mathcal{F} \text{ or } A^c \in \mathcal{F}.$$

c) If X is infinite there are *nontrivial ultrafilter* $\mathcal{U}$, i.e. with the property that $\mathcal{U}$ does not contain finite set (or equivalently, (why?) singletons).

d) Let $x \in \ell_\infty$ and let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. Then there an $r = r(\mathcal{U}, x) \in \mathbb{R}$, so that for all $\varepsilon > 0$ there is an $N \in \mathcal{U}$ so that $|x_n - r| < \varepsilon$ for all $n \in N$.

e) Think of an ultrafilter to be a directed set (reversed inclusion) and pick for every $N \in \mathcal{U}$ and element $k_N \in \mathbb{N}$. Then for all $x = (x_n) \in \ell_\infty$

$$r(\mathcal{U}, x) = \lim_{U \in \mathcal{U}} x_{k_U}.$$

f) Show that for every ultrafilter the map

$$\mathcal{U}(\cdot) : X \to \mathcal{F}, \quad x \to r(\mathcal{U}, x)$$

is bounded and linear (and thus an element of ℓ_∞^*).

Proof (a) Let $\mathcal{F}_0$ be filter on A and define

$$\mathcal{E} = \{\mathcal{F} : \mathcal{F} \text{ is a filter on } A \text{ and } \mathcal{F}_0 \subset \mathcal{F}\}.$$

Then it is easy to see (but needs to be shown) that $\mathcal{E}$ satisfies (with respect to $\subset$) the Hausdorff Maximal Principle and contains a maximal element, which has to be (by definition of ultrafilter) an ultrafilter.

(b) $\Rightarrow$ Let $\mathcal{U}$ be a ultra filter, and let $A \subset X$. Then we claim that either

$$\mathcal{U} \cap A = \{B \subset X : \exists F \in \mathcal{U}\, F \cap A \subset B : F \in \mathcal{U}\}$$

or

$$\mathcal{U} \cap A^c = \{B \subset X : \exists F \in \mathcal{U}\, F \cap A^c \subset B : F \in \mathcal{U}\}$$

is again a filter. Indeed if $A \cap F \not\emptyset$for all $F \in \mathcal{F}$, then $\mathcal{U} \cap A$ is a filter, if $A^c \cap F \not\emptyset$for all $F \in \mathcal{F}$, then $cU \cap A^c$ is a filter. One of this cases must occur, otherwise for some $F_1, F_2 \in \mathcal{U}$

$$F_1 \cap F_2 = (F_1 \cap A^c) \cup (F_2 \cap A) = \emptyset.$$

Now if w.l.o.g. $\mathcal{U} \cap F$ is a filter it must follow from the maximality of $\mathcal{U}$ that $A \in \mathcal{U}$.

$\Leftarrow$ If left hand condition is true and $A \notin \mathcal{F}$, and thus $A^c \in \mathcal{F}$, then $\mathcal{F}$ cannot be contained in a filter $\mathcal{F}'$ which also contains A, otherwise $\emptyset = A \cap A^c \in \mathcal{F}'$.

c) Consider $\mathcal{F}_0 := \{F : F^c \text{ is finite }\}$. Note that $\mathcal{F}_0$ is a filter. Thus our claim follows from (a).

d) We assume w.l.o.g. that $\mathcal{F} = \mathbb{R}$ (for $\mathbb{C}$ the proof is similar). Let $\mathcal{U}$ be an Ultrafilter on $\mathbb{N}$. From (b) it follows by induction on $c = 1, 2, 3, \ldots$ that if $(N_1, N_2, \ldots N_c)$ is a partition of $\mathbb{N}$, there must be a $j \in \{1, 2 \ldots c\}$ so that $N_j \in \mathcal{U}$. This implies that there must be closed intervals I_j,

$$[-\|x\|, \|x\|] = I_0 \supset I_1 \supset I_2 \ldots,$$

whose lengths converges to 0 an so that

$$N_j = \{n \in \mathbb{N} : x_n \in I_j\} \in \mathcal{U}.$$

By compactness the must be a unique point $r(\mathcal{U}, x)$ so that

$$\{r(\mathcal{U}, x)\} = \bigcap_{j=1}^{\infty} I_j.$$

this number will do the job.
e) Follows easily from (d).
(f) Linearity follows easily from (e), and boundedness follows since $r(x\mathcal{U}) \in [-\|x\|, \|x\|]$ (see proof of (d).

Problem 6. Let X be locally convex space over $\mathbb{R}$, $A \subset X$ closed and convex and $K \subset X$ compact and convex, and assume that A and K are disjoint and both non empty. Show that there is an $f \in X^*$ so that

$$\sup_{x\in A} f(x) < \inf_{x\in K} f(x).$$

Give an example which shows that one cannot replace K compct by only K closed (of course all other conditions are satisfied).

Proof. Let $\mathcal{N}_0$ be the open convex neighborhoods of 0. First we will show that there is an $U \in \mathcal{N}_0$ so that $K + U$ and F are disjoint.

Indeed, assuming that this were not true, we pick and $x_U \in (A \cap (U + K))$ and write $x_U = y_U + z_U$ with $y_U \in U$ and $z_U \in K$. Now $(y_U)_{U\in\mathcal{N}_0}$ is net (reversed inclusion) which converges to 0, Since K is compact (z_U) has a convergent subnet, say $(z'_i)_{i\in I}$ which converges to some $z \in K$ (recall I is a directed set and there is a map $J : I\mathcal{N}_0$ with $z'_i = z_{J(i)}$ and so that for all $U \in \mathcal{N}_0$ there is a $i_0 \in$, so that $J(i) \subset U$ for all $i \geq i_0$). Let $x'_i := x_{J(i)}$, then (x'_i) also converges to z. But this implies that $z \in K \cap A$ which is a contradiction.

Now the proof follows like n class were we proved the claim for K being a singleton. It is easy to see that $K + U$ (U as chosen above) is open (thus has internal points) and convex. After a shift we can assume that $0 \in K$.

By the Geometrical Hahn Banach Theorem we can find a linear non zero map $f : X \to X$ so that $\inf_{x\in\ K+U} \geq \sup_{x\in A}$. This implies that $\alpha = \sup_{x\in A} \leq 0$ ($0 \in K + U$!). f is continuous, since it is bounded on $U \cap (-U)$.

We claim that $\min_{x\in K} f(x) > \alpha$. Indeed, if for some $x \in K$ $f(x) = \alpha$, we could find a $z \in U$ so that $f(z) < 0$ ($f \not\equiv 0$ and U open) but then $f(x + z) < \alpha$) which is a contradiction.

Example: Take $X = \mathbb{R}^2$, $A = \{(x, y) : x > 0 \text{ and} y \geq 1/x\}$, and $B = \{(x, y) : x > 0 \text{ and} y \leq -1/x\}$. Then

$$\operatorname{dist}(A, B) = \inf_{(x,y)\in A, (\tilde{x},\tilde{y})\in B} \|(x, y) - (\tilde{x}, \tilde{y})\| \leq \lim_{t\to\infty} \left|\frac{2}{t}\right| = 0,$$

which implies a strictly separating continuous functional does not exist.

Problem 7. 45/page 170.

The space $C_\infty(\mathbb{R})$ of all infinitely often differentiable functions on $\mathbb{R}$ has a Frèchet space topology for which

$$f_n \to f \iff \forall k \in \mathbb{N}_0,\ f^(k)_n \to_{n\to\infty} f^{(k)} \text{ uniformly on compacta..}$$

Proof. For $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Define

$$p_{(k,m)} : C_\infty(\mathbb{R}) \to [0\infty), \quad p_{(k,m)}(f) := \sup_{x\in[-m,m]} |f^{(k)}(x)|.$$

Then verify that $(p_{(k,m)})$ are a (countable) family of semi norms thatsatisfy the claim.

Problem 8. 49/page 170. Assume that X is an infinite dimensional Banach space.

a) Show that weak open sets in X and weak* open sets in X^* are unbounded.
b) Bounded sets in X (and X^*) are meager with respect to the weak (and weak*) topology.
c) X is meager with respect to the weak, and X^* is meager with respect to the weak* topology.
d) There is no translation invariant metric on X^* which generates the weak* topology.

Proof. a) Follows from the fact (which was shown for Problem 5/Homework 6) that for $f_1, f_2, \dots f_n \in X^*$, it follows that

$$Y = \bigcap_{j=1}^{n} f_j^{\perp} = \bigcap_{j=1}^{n} \{x \in X : f_j(x) = 0\},$$

is an infinite dimensional subspace.

b) Every bounded set is a subset of some closed ball having large enough radius around 0. By Problem 6/Homework 6 or using that "weakly closed" is equivalent to "norm closed" for convex sets (proven in class) we deuce that closed balls in X are weakly closed, and by Alaoglu - Bourbaki B_{X^*} is weak* compact and thus closed. But it follows from part (a) that the weak open kernel of any ball in X as well the weak* open kernel of any ball in X^* is empty

c) $X = \bigcup_{n\in\mathbb{N}} nB_X$ and $X^* = \bigcup_{n\in\mathbb{N}} nB_{X^*}$.

d) If there were a translation invariant metric generating the weak*-topology this metric d would turn (X^*, d) into a complete metric space, then a net in X^* is weak* Cauchy net (see page 167) if and only if it is a d-Cauchy net (needs verification).

But then (X^*, weak^*) would be a complete metric space, which contradicts part (c) and the Baire Category Theorem.