Problems in Real Variables, II (Math608), Solutions
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Problem 1.

a) Show that the closure of each convex set in a topological vector space is convex.

b) Show that in $\mathbb{R}^n$ each internal point of a convex set is an interior point.

c) Give an example of a subset of $\mathbb{R}^2$ which has an internal point which is not in the interior of that set.

Proof. a) Let $C$ be convex and let $x, y \in \overline{C}$ and $\lambda \in [0, 1]$. Then there are nets $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ in $C$ (take for example $I$ to be a neighborhood basis of 0 and pick for every $U \in I$ elements $x_U \in (x + U) \cap C$ and $y_U \in (y + U) \cap C$). Then

$$\lambda x + (1 - \lambda) y = \lim_{i \in I} \lambda x_i + (1 - \lambda) y_i \in \overline{C}.$$ 

b) Let $C \subset \mathbb{R}^n$ be convex, and assume that $x$ is an internal point of $C$. For each vector $u = (u_1, u_2, \ldots u_n)$ with $u_i = \pm 1$ there is $\delta(u) > 0$ so that $x + ru \in C$ for all $0 \leq r \leq \delta(u)$. Choose

$$\delta_0 = \min_{u \in \{-1,1\}^n} \delta(u).$$

Then $Q = x + \text{conh}\{\delta_0 u, u \in \{-1,1\}^n\}$ is an $n$-dimensional cube with $x$ as center, with positive sidelength $a_0$ which is contained in $C$. Thus $x \in Q^0 \subset C^0$.

c) Let $U$ be the set of all unit vectors in $\mathbb{R}^n$. Choose a sequence of unit vectors $(u_n)$ and consider the set

$$A = \bigcup_{u \in U \setminus \{u_n, n \in \mathbb{N}\}} [-u, u] \cup \bigcup_{n=1}^{\infty} \left[ -\frac{1}{n} u_n, \frac{1}{n} u_n \right].$$

Then $0$ is an internal but not interior point of $A$.

Problem 2. Show that $B_{L_1[0,1]}$ does not have any extreme points. Deduce from this that $L_1[0,1]$ is not isometric to the dual of a Banach space.

Let $f \in B_{L_1[0,1]}$. We need to show that $f$ is not extreme point of $B_{L_1[0,1]}$. If $\|f\|_1 < 1$ we simply write $f$ as a non trivial convex combination as follows

$$f = \|f\| \frac{f}{\|f\|} + (1 - \|f\|)0.$$ 

If $\|f\| = 1$ there must be a $\varepsilon > 0$ so that w.l.o.g (after possibly switching to $-f$ if necessary)

$$\lambda(A) \geq \varepsilon \text{ with } A = \{x \in [0,1] : f(x) \geq \varepsilon\}$$
and we can therefore split \( A \) into sets \( A_+ \) and \( A_- \) with \( m(A_+) = m(A) = \frac{1}{2}m(A) \) and write
\[
f = \frac{1}{2} \left( f + \varepsilon \chi_{A_+} - \varepsilon \chi_{A_-} \right) + \left( f - \varepsilon \chi_{A_+} + \varepsilon \chi_{A_-} \right).
\]

**Problem 3.** Determine the extreme points of the unit balls of \( c_0 \) and \( \ell_1 \) and \( \ell_\infty \). Deduce that \( c_0 \) is not the dual of a some Banach space \( X \).

**Proof.** Assume \( x = (x_n) \in B_{c_0} \) then there is an \( n_0 \) so that \( |x_{n_0}| < \frac{1}{2} \), thus \( x \pm \frac{1}{2}e_{n_0} \in B_{c_0} \) and
\[
x = \frac{1}{2} (x + \frac{1}{2} e_{n_0}) + \frac{1}{2} (x - \frac{1}{2} e_{n_0}).
\]

Thus we showed that every \( x \in B_{c_0} \) can be written as convex combination of to other elements of \( B_{c_0} \). It follows therefore that \( \text{Ext}(B_{c_0}) = \emptyset \).

We claim that
\[
\text{Ext}(B_{\ell_\infty}) = \{ (x_n) : |x_n| = 1 \text{ for } n \in \mathbb{N} \}.
\]

Indeed, if \( x \in B_{\ell_\infty} \) has (at least one coordinate \( x_{n_0} \), with \( |x_{n_0}| = 1 - \varepsilon < 1 \) we can write (similarly as before)
\[
x = \frac{1}{2} (x + \varepsilon e_{n_0}) + \frac{1}{2} (x - \varepsilon e_{n_0}),
\]
as convex combination of to other elements in \( B_{\ell_\infty} \).

In order to show that every \( (x_n) \in \ell_\infty \), with \( |x_n| = 1 \), \( n \in \mathbb{N} \), is an extreme point of \( B_{\ell_\infty} \). We first observe that following

**Claim.** If we write a number \( x \in \mathbb{F} \), with \( |x| = 1 \) as a convex combination, with of two different numbers \( y \) and \( z \), in \( \mathbb{F} \) either \( |y| > 1 \) or \( |z| > 1 \).

The claim is trivial for \( \mathbb{F} = \mathbb{R} \), and not that hard to see in \( \mathbb{F} = \mathbb{C} \) (if a point on the circle is the convex combination of two different points one of them must be outside of the circle).

Thus if \( x = \lambda y + (1 - \lambda)z \), with \( x = (x_n) \), \( |x_n| = 1 \), for \( n \in \mathbb{N} \), \( \lambda \in (0,1) \) and \( y = (y_n) \neq z = (z_n) \), for some \( n_0 \), \( x_{n_0} \neq z_{n_0} \) and then either \( |y_{n_0}| \) or \( |z_{n_0}| \), must be larger than 1.

For \( \ell_1 \) we have \( \text{Ext}(B_{\ell_1}) = \{ \pm e_n : n \in \mathbb{N} \} \). Indeed \( \subset \) follows from above claim again. And \( \supset \) follows from the fact that if \( x = (x_n) \in B_{\ell_1} \) \( \setminus \{ \pm e_n : n \in \mathbb{N} \} \), and \( |x| = 1 \) there must be \( m \neq n \) in \( \mathbb{N} \) \( 0 < \varepsilon = \min(|x_m|, |x_n|) \leq \max(|x_m|, |x_n|) < 1 \), and thus we write
\[
x = \frac{1}{2} ((x + \varepsilon \text{sign}(x_m) e_m - \text{sign}(x_n) e_n) + (x - \varepsilon \text{sign}(x_m) e_m + \text{sign}(x_n) e_n)).
\]

If \( |x| < 1 \) we simply write
\[
x = |x| \frac{x}{|x|} + (1 - |x|)0.
\]

**Problem 4.** Determine the extreme points of \( C[0,1] \).

**Hint:** The answer for \( \mathbb{F} = \mathbb{R} \) and \( \mathbb{F} = \mathbb{C} \) will be (very) different.
**Proof.** For $F = \mathbb{R}$ as well as $F = \mathbb{C}$ we can write
\[
ExtB_{C[0,1]} = \{ f : [0, 1] \to F : f \text{ continuous and } |f(t)| = 1 \text{ for all } t \in [0, 1] \}.
\]
Note that this set only consists of the two constant functions $f \equiv \pm 1$ if $F = \mathbb{R}$, while for $F = \mathbb{C}$ it can be actually shown (but not that easy) that
\[
B_{C[0,1]} = \text{convh}(Ext(B_{C[0,1]})) = \text{convh}(\{ f : [0, 1] \to \mathbb{C} : f \text{ continuous and } |f(t)| = 1 \text{ for all } t \in [0, 1] \}).
\]

To prove our claim we first show that if $|f| \neq 1$, then $f$ is not an extreme point of $B_{C[0,1]}$. If $|f(t_0)| < 1$ for some $t_0 \in [0, 1]$, then, by continuity, there is an open non empty interval $I = [a, b] \cap \mathbb{R}$ and an $\varepsilon > 0$ so that $b - a \geq \varepsilon$ and $|f(t)| \leq 1 - \varepsilon$ for all $t \in I$. Let $g$ we a function on $[0, 1]$ with $g|_{[0,1]\setminus I} \equiv 0$, $g((a+b)/2) = \varepsilon$ and $\|g\|_u = \varepsilon$, Then $f \pm g \in B_{C[0,1]}$ and
\[
f = \frac{1}{2}((f + g) + (f - g)).
\]

Let $f \in B_{C[0,1]}$, so that $|f(t)| = 1$ for all $t \in [0, 1]$. In order to show that $f$ is an extreme point of $B_{C[0,1]}$. We write $f = \lambda h + (1 - \lambda)g$, with $h, g \in C[0,1]$, and assume that this linear combination is not trivial, i.e. $0 < \lambda < 1$, and $h \neq g$. There must be some $t_0 \in [0, 1]$ so that $h(t_0) \neq g(t_0)$, since $f(t_0) = \lambda h(t_0) + (1 - \lambda)g(t_0)$ it follows form above claim that $|g(t_0)| > 1$ or $|h(t_0)| > 1$, thus $g$ and $h$ cannot be both in $B_{C[0,1]}$.

**Problem 5.** Let $X$ be locally convex space over $\mathbb{R}$, $A \subset X$ closed and convex and $K \subset X$ compact and convex, and assume that $A$ and $K$ are disjoint and both non empty. Show that there is an $f \in X^*$ so that
\[
\sup_{x \in A} f(x) < \inf_{x \in K} f(x).
\]
Give an example which shows that one cannot replace $K$ compact by only $K$ closed (of course all other conditions are satisfied).

**Proof.** We first need to show that there is a convex open neighborhood $U$ of 0, so that $(K + U) \cap F = \emptyset$. Then the proof works as in the case that $K = \{x_0\}$ which was done in class.

Assume that for every open neighborhood $U$ one can choose an $x_U \in (K + U) \cap F$ write $x_U = z_U + y_U$ with $z_U \in K$ and $y_U \in U$. Then the net $(y_U)_{U \in \mathcal{N}_0}$ ($\mathcal{N}_0$ set of open neighborhoods of 0) is a net which converges to 0 since $(z_U) \subset K$, there is a convergent subnet $(z_i)_{i \in I}$ which converges to some $z \in K$, thus $(z_i + y_i)_{i \in I}$ converges also to $z$ which also has to be in $F$. But this is a contradiction.

**Problem 6.** Let $(f_n)$ be a sequence of continuous functions on $[0, 1]$ such that, for each $x \in [0, 1]$ there is an $n(x)$ such that $f_n(x) \geq 0$ for all $n \geq n(x)$.

Prove that there is an open, not empty interval $I \subset [0, 1]$ and an $N \in \mathbb{N}$ so that $f_n(x) \geq 0$ for all $x \in I$ and $n \geq N$. 
Sketch. Straightforward application of the Baire Category Theorem applied to the (closed sets)
\[ F_n := \bigcap_{k \geq n} \{ x \in [0,1] : f_k(x) \geq 0, \text{ for all } k \geq n \}. \]

Problem 7. Let $X$ be a Banach space and $Y$ a closed subspace of $X$. Show that every extreme point of the unit ball of the dual $Y^*$ extends to an extreme point of the unit ball of $X^*$.

Proof. Let $f^* \in \text{Ext}(B_{Y^*})$ and consider
\[ C_{f^*} = \{ F \in X^* : \|F\| \leq 1, F|_Y = f \}. \]
It is easy to verify that $C_{f^*}$ convex $w^*$-compact and not empty (by Hahn Banach). Thus by the Krein Milman Theorem $C_{f^*}$ must contain an extreme point $F^*$.

We claim that $F^*$ is an extreme point of $B_{X^*}$.

Indeed, write
\[ F^* = \lambda G^* + (1 - \lambda)H^*, \]
with $0 < \lambda < 1$, $G^*, H^* \in B_{X^*}$. Since $f = \lambda G^*|_Y + (1 - \lambda)H^*|_Y$, follows first that $G^*|_Y = H^*|_Y = f$, since $f$ is an extreme point of $B_{Y^*}$. Thus $F^*, G^*, H^* \in C_{f^*}$. Since $F^*$ is extreme point of $C_{f^*}$ it follows that $H^* = G^* = F^*$. 