

The lattice of closed ideals in the Banach algebra of operators on a certain dual Banach space

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Abstract

In this paper we determine the closed operator ideals of the space $F := \left(\bigoplus_{n \in \mathbb{N}} \ell_2^n \right)_{\ell_1}$.

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1 Introduction

The aim of this note is to classify the closed ideals in the Banach algebra $\mathcal{B}(F)$ of (bounded, linear) operators on the Banach space

$$F := \left(\bigoplus_{n \in \mathbb{N}} \ell_2^n \right)_{\ell_1}. \quad (1.1)$$

More precisely, we shall show that there are exactly four closed ideals in $\mathcal{B}(F)$, namely $\{0\}$, the compact operators $\mathcal{K}(F)$, the closure $\overline{\mathcal{G}}_{\ell_1}(F)$ of the set of operators factoring through ℓ_1 , and $\mathcal{B}(F)$ itself.

The collection of Banach spaces E for which a classification of the closed ideals in $\mathcal{B}(E)$ exists is very sparse. Indeed, the following list appears to be the *complete* list of such spaces.

- (i) For a finite-dimensional Banach space E , $\mathcal{B}(E) \cong M_n$, where n is the dimension of E , and so it is ancient folklore that $\mathcal{B}(E)$ is simple in this case.
- (ii) In 1941 Calkin [2] classified all the ideals in $\mathcal{B}(\ell_2)$. In particular he proved that there are only three closed ideals in $\mathcal{B}(\ell_2)$, namely $\{0\}$, $\mathcal{K}(\ell_2)$, and $\mathcal{B}(\ell_2)$.
- (iii) In 1960 Gohberg, Markus, and Feldman [5] extended Calkin's theorem to the other classical sequence spaces. More precisely, they showed that $\{0\}$, $\mathcal{K}(E)$, and $\mathcal{B}(E)$ are the only closed ideals in $\mathcal{B}(E)$ for each of the spaces $E = c_0$ and $E = \ell_p$, where $1 \leq p < \infty$.

- (iv) Later in the 1960'ies Gramsch [6] and Luft [9] independently extended Calkin's theorem in a different direction by classifying all the closed ideals in $\mathcal{B}(H)$ for each Hilbert space H (not necessarily separable). In particular, they showed that these ideals are well-ordered by inclusion.
- (v) In 2003 Laustsen, Loy, and Read [7] proved that, for the Banach space

$$E := \left(\bigoplus_{n \in \mathbb{N}} \ell_2^n \right)_{c_0}, \quad (1.2)$$

there are exactly four closed ideals in $\mathcal{B}(E)$, namely $\{0\}$, the compact operators $\mathcal{K}(E)$, the closure $\overline{\mathcal{G}}_{c_0}(E)$ of the set of operators factoring through c_0 , and $\mathcal{B}(E)$ itself.

Note that the Banach space F given by (1.1) is the dual of the Banach space E given by (1.2), and so the result of this note can be seen as a 'dualization' of [7]. In fact, our strategy draws heavily on the methods introduced in [7].

2 The classification theorem

We begin this section by recalling various definitions and results from [7]. For simplicity we state the results only in the generality that is required for our present purposes, but emphasize that a number of them hold true in greater generality.

2.1 ℓ_1 -direct sums. Let (E_n) be a sequence of Banach spaces. We denote by $(\bigoplus E_n)_{\ell_1}$ the ℓ_1 -direct sum of E_1, E_2, \dots , that is, the collection of sequences (x_n) such that $x_n \in E_n$ for each $n \in \mathbb{N}$ and

$$\|(x_n)\| := \sum_{n=1}^{\infty} \|x_n\| < \infty. \quad (2.1)$$

This is a Banach space for coordinatewise defined addition and scalar multiplication and norm given by (2.1).

Set $E := (\bigoplus E_n)_{\ell_1}$. For each $m \in \mathbb{N}$, we write J_m^E for the canonical embedding of E_m into E and Q_m^E for the canonical projection of E onto E_m . Both J_m^E and Q_m^E are operators of norm one; in fact, the former is an isometry, and the latter is a quotient map. When no ambiguity may arise, we omit the superscript E from the operators J_m^E and Q_m^E .

We use similar notation and conventions for finite collections of Banach spaces and operators.

2.2 Definition. Let (E_n) and (F_n) be sequences of Banach spaces, and let $T: (\bigoplus E_n)_{\ell_1} \rightarrow (\bigoplus F_n)_{\ell_1}$ be an operator. We associate with T the infinite matrix $(T_{m,n})$, where

$$T_{m,n} := Q_m^F T J_n^E: E_n \rightarrow F_m \quad (m, n \in \mathbb{N}).$$

The *support* of the n^{th} column of T is

$$\text{colsupp}_n(T) := \{m \in \mathbb{N} \mid T_{m,n} \neq 0\} \quad (n \in \mathbb{N}).$$

We say that T has *finite columns* if each column has finite support, and in this case we set $\mu_n(T) := \max(\text{colsupp}_n(T))$.

The significance of operators with finite columns lies in the fact that, in the case where each of the spaces E_n ($n \in \mathbb{N}$) is finite-dimensional, for each operator $T: (\bigoplus E_n)_{\ell_1} \rightarrow (\bigoplus F_n)_{\ell_1}$ we can find a perturbation $\tilde{T}: (\bigoplus E_n)_{\ell_1} \rightarrow (\bigoplus F_n)_{\ell_1}$ with finite columns such that the difference $T - \tilde{T}$ is compact and has arbitrarily small norm (see [7, Lemma 2.7(i)]).

2.3 Diagonal operators. Let (E_n) and (F_n) be sequences of Banach spaces, and, for each $n \in \mathbb{N}$, let $T_n: E_n \rightarrow F_n$ be an operator. Suppose that $\sup \|T_n\| < \infty$. Then we can define the *diagonal operator*

$$\text{diag}(T_n): (x_n) \mapsto (T_n x_n), \quad \left(\bigoplus E_n\right)_{\ell_1} \rightarrow \left(\bigoplus F_n\right)_{\ell_1}.$$

Clearly, we have $\|\text{diag}(T_n)\| = \sup \|T_n\|$.

The following construction is a dual version of [7, Construction 4.2].

2.4 Construction. Let (E_n) and (F_n) be sequences of Banach spaces, and set $E := (\bigoplus E_n)_{\ell_1}$ and $F := (\bigoplus F_n)_{\ell_1}$. Further, set $\tilde{F} := (\bigoplus \tilde{F}_n)_{\ell_1}$, where $\tilde{F}_n := F$ for each $n \in \mathbb{N}$. Let $T: E \rightarrow F$ be an operator. Since $\|T J_n^E\| \leq \|T\|$ for each $n \in \mathbb{N}$, we have a diagonal operator $\text{diag}(T J_n^E): E \rightarrow \tilde{F}$.

Define

$$W: \tilde{F} \rightarrow F, \quad y = (y_n)_{n \in \mathbb{N}} \mapsto \left(\sum_{m=1}^{\infty} Q_m^F y_m \right)_{m \in \mathbb{N}}.$$

Then it follows that $\|W\| \leq 1$ and T can be written as

$$T = W \text{diag}(T J_n^E). \tag{2.2}$$

2.5 Definition. Let G be a closed subspace of a Hilbert space H . We denote by G^\perp the orthogonal complement of G in H , and write proj_G for the orthogonal projection of H onto G (so that proj_G is the idempotent operator on H with image G and kernel G^\perp). Let $m \in \mathbb{N}$, let E be a Banach space, and let K_1, \dots, K_m be Hilbert spaces. For each operator $T: E \rightarrow (K_1 \oplus \dots \oplus K_m)_{\ell_1}$ and each $\varepsilon \geq 0$, set

$$n_\varepsilon(T) := \sup \left\{ n \in \mathbb{N}_0 \mid \begin{array}{l} \|(\text{proj}_{G_1^\perp} \oplus \dots \oplus \text{proj}_{G_m^\perp})T\| > \varepsilon \\ \text{whenever } G_j \subset K_j \text{ are subspaces} \\ \text{with } \dim G_j \leq n, \text{ for } j = 1, \dots, m \end{array} \right\} \in \mathbb{N}_0 \cup \{\pm\infty\}.$$

2.6 Lemma. Let $m \in \mathbb{N}$, let H and K_1, \dots, K_m be Hilbert spaces, let $T: H \rightarrow (K_1 \oplus \dots \oplus K_m)_{\ell_1}$ be an operator, and let $0 < \varepsilon < \|T\|$.

- (i) Suppose that $n_\varepsilon(T)$ is finite. Then there are operators $R: H \rightarrow \ell_1$ and $S: \ell_1 \rightarrow (K_1 \oplus \dots \oplus K_m)_{\ell_1}$ such that $\|T - SR\| \leq \varepsilon$, $\|R\| \leq \|T\| \sqrt{n_\varepsilon(T) + 1}$, and $\|S\| \leq 1$.

- (ii) For each natural number $n \leq n_\varepsilon(T)/2 + 1$, there are operators $U: \ell_2^n \rightarrow H$ and $V: (K_1 \oplus \dots \oplus K_m)_{\ell_1} \rightarrow \ell_2^n$ such that $I_{\ell_2^n} = VTU$, $\|U\| \leq 1/\varepsilon$, and $\|V\| \leq 1$.
- (iii) Let $k \in \mathbb{N}$, let H_0 be a closed cofinite subspace of H , and suppose that $n_\varepsilon(T) \geq \dim H_0^\perp + k$. Then $n_\varepsilon(T|_{H_0}) \geq k$.

The proofs of (i) and (ii) of Lemma 2.6 follow along the same line as the proof of [7, Lemma 5.3(i)–(ii)]. Actually we could deduce (i) and (ii) by applying [7, Lemma 5.3(i)–(ii)] to some variant of the adjoint of T . For the sake of being selfcontained we include a proof.

Proof. (i). Let $n = n_\varepsilon + 1$ and choose subspaces $G_j \subset K_j$, $\dim G_j \leq n$, for $j = 1, 2, \dots, m$, so that

$$\|(\text{proj}_{G_1^\perp} \oplus \text{proj}_{G_2^\perp} \oplus \dots \oplus \text{proj}_{G_m^\perp}) \circ T\| = \sum_{j=1}^m \|\text{proj}_{G_j^\perp} Q_j T\| \leq \varepsilon.$$

We put $d = \sum_{j=1}^m \dim G_j$. For $j = 1, \dots, m$ we let $I_j: G_j \rightarrow \ell_2^{\dim G_j}$ be an isometry and $\tilde{I}_j: \ell_2^{\dim G_j} \rightarrow \ell_1^{\dim G_j}$ the formal identity. For $x \in H$ we define

$$R(x) = (\tilde{I}_j I_j \text{proj}_{G_j} Q_j T(x))_{j=1}^m \in (\oplus_{j=1}^m \ell_1^{\dim G_j})_{\ell_1} = \ell_1^d,$$

and for $z = (z_j)_{j=1}^m \in (\oplus_{j=1}^m \ell_1^{\dim G_j})_{\ell_1}$ we define

$$S(z) = (I_j^{-1} \tilde{I}_j^{-1}(z_j))_{j=1}^m \in (\oplus_{j=1}^m K_j)_{\ell_1}.$$

Since $\|\tilde{I}_j\| \leq \sqrt{\dim G_j} \leq \sqrt{n}$ and $\|\tilde{I}_j^{-1}\| \leq 1$ it follows that $\|R\| \leq \|T\|\sqrt{n}$ and $\|S\| \leq 1$. Furthermore, $T - SR = (\text{proj}_{G_1^\perp} \oplus \text{proj}_{G_2^\perp} \oplus \dots \oplus \text{proj}_{G_m^\perp}) \circ T$, which proves (i).

(ii). Let $n \leq \frac{1}{2}n_\varepsilon(T) + 1$. By induction we will choose normalized elements $x_1, x_2, \dots, x_n \in H$ so that

- a) $x_i \perp x_{i'}$, whenever $i \neq i'$ are in $\{1, 2, \dots, n\}$,
- b) $T(x_i) \perp T(x_{i'})$, whenever $i \neq i'$ are in $\{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, m\}$,
- c) $\|T(x_i)\| = \sum_{j=1}^m \|Q_j T(x_i)\| > \varepsilon$, whenever $i \in \{1, 2, \dots, n\}$.

Choose $x_1 \in H$, $\|x_1\| = 1$ so that $\|T(x_1)\| \geq \varepsilon$, and assuming that x_1, x_2, \dots, x_{k-1} were chosen for some $k \leq n$, we put for $j = 1, \dots, m$

$$G_j = \text{span}\{Q_j T(x_i), Q_j T T^* T x_i : i = 1, \dots, k-1\} \subset K_j.$$

Since $\dim G_j \leq 2k - 2 \leq 2n \leq n_\varepsilon(T)$, there is a $w \in H$, $\|w\| = 1$, so that

$$\|(\text{proj}_{G_1^\perp} \oplus \dots \oplus \text{proj}_{G_m^\perp}) T(w)\| > \varepsilon.$$

We write

$$w = \sum_{i=1}^k \alpha_i x_i + \beta T^* T x_i + z, \text{ with } z \in \text{span}^\perp\{x_i, T^* T x_i : i = 1, \dots, k\}.$$

Since $\text{proj}_{G_j^\perp} Q_j T(x_i) = \text{proj}_{G_j^\perp} Q_j T T^* T x_i = 0$, it follows that $(\text{proj}_{G_1^\perp} \oplus \dots, \text{proj}_{G_m^\perp}) T(w) = (\text{proj}_{G_1^\perp} \oplus \dots, \text{proj}_{G_m^\perp}) T(z)$ and, thus, $x_k = z/\|z\|$ satisfies the conditions (a), (b) and (c).

Finally we define

$$U : \ell_2^n \rightarrow H, \quad e_i \mapsto \frac{x_i}{\varepsilon}, \text{ for } i = 1, \dots, n$$

and

$$V : \left(\bigoplus_{j=1}^m K_j \right)_{\ell_1} \rightarrow \ell_k, \quad (y_j)_{j=1} \mapsto \sum_{j=1}^m \sum_{i=1}^n e_i \frac{\langle y_j, Q_j T(x_i) \rangle \varepsilon}{\|Q_j T(x_i)\| \sum_{\ell=1}^m \|Q_\ell T(x_i)\|}.$$

Because of (a) we have $\|U\| \leq 1/\varepsilon$, and using the triangle inequality and (c) it is easy to see that $\|V\| \leq 1$ and using (b) we deduce that $I_{\ell_2^n} = VTU$.

(iii). For each $j = 1, \dots, m$, let G_j be a subspace of K_j with $\dim G_j \leq k$. Set $F_j := G_j + Q_j T(H_0^\perp)$. Then F_j is finite-dimensional with $\dim F_j \leq n_\varepsilon(T)$, and so we can find $x \in H$ such that $\|x\| \leq 1$ and $\|(\text{proj}_{F_1^\perp} \oplus \dots \oplus \text{proj}_{F_m^\perp}) T x\| > \varepsilon$. It follows that

$$\begin{aligned} \|(\text{proj}_{G_1^\perp} \oplus \dots \oplus \text{proj}_{G_m^\perp}) T|_{H_0}\| &\geq \|(\text{proj}_{G_1^\perp} \oplus \dots \oplus \text{proj}_{G_m^\perp}) T(\text{proj}_{H_0} x)\| \\ &\geq \|(\text{proj}_{F_1^\perp} \oplus \dots \oplus \text{proj}_{F_m^\perp}) T(\text{proj}_{H_0} x)\| \\ &= \|(\text{proj}_{F_1^\perp} \oplus \dots \oplus \text{proj}_{F_m^\perp}) T x\| > \varepsilon, \end{aligned}$$

and so $n_\varepsilon(T|_{H_0}) \geq k$. □

2.7 Remark. Let (K_n) be a sequence of Hilbert spaces, and let T be an operator on $(\bigoplus K_n)_{\ell_1}$ with finite columns. As in [7, Remark 5.4], there is a natural way to define $n_\varepsilon(TJ_m)$ for each $\varepsilon \geq 0$ and each $m \in \mathbb{N}$, namely by ignoring the cofinite number of Hilbert spaces K_k such that $Q_k T J_m = 0$.

For each pair (E, F) of Banach spaces, set

$$\mathcal{G}_{\ell_1}(E, F) := \{TS \mid S \in \mathcal{B}(E, \ell_1), T \in \mathcal{B}(\ell_1, F)\}.$$

The fact that $\ell_1 \cong \ell_1 \oplus \ell_1$ implies that \mathcal{G}_{ℓ_1} is an operator ideal, and so its closure $\overline{\mathcal{G}}_{\ell_1}$ is a closed operator ideal. As usual, we write $\overline{\mathcal{G}}_{\ell_1}(E)$ instead of $\overline{\mathcal{G}}_{\ell_1}(E, E)$.

2.8 Lemma. Let E be a Banach space and \mathcal{J} be an ideal in $\mathcal{B}(E)$. If P is an idempotent operator on E and $P \in \overline{\mathcal{J}}$, then in fact $P \in \mathcal{J}$.

Proof. Let (T_n) be a sequence in \mathcal{J} converging to P . Replacing T_n with PT_nP we may assume that $T_n \in P\mathcal{B}(E)P$ for all $n \in \mathbb{N}$. Note that $P\mathcal{B}(E)P$ is a Banach algebra with unit P , and so there exists n such that T_n is invertible. Thus there is an operator $U \in \mathcal{B}(E)$ with $P = (PUP)T_n$, which implies that $P \in \mathcal{J}$. □

2.9 Theorem. Set $F := (\bigoplus \ell_2^n)_{\ell_1}$. For each operator T on F with finite columns the following conditions are equivalent:

- (i) $T \notin \overline{\mathcal{G}}_{\ell_1}(F)$,
- (ii) $\sup\{n_\varepsilon(TJ_k) \mid k \in \mathbb{N}\} = \infty$ for some $\varepsilon > 0$,
- (iii) there are operators U and V on F such that $VTU = I_F$.

Proof. We begin by proving the implication “not (ii) \Rightarrow not (i)”

Let $0 < \varepsilon < \|T\|$, and suppose that $c := \sup\{n_\varepsilon(TJ_k) \mid k \in \mathbb{N}\} < \infty$. Then, for each $k \in \mathbb{N}$, there are operators $R_k: \ell_2^k \rightarrow \ell_1$ and $S_k: \ell_1 \rightarrow F$ such that $\|TJ_k - S_kR_k\| \leq \varepsilon$, $\|R_k\| \leq \|T\| \sqrt{c+1}$, and $\|S_k\| \leq 1$ by Lemma 2.6(i). In the notation of Construction 2.4 (with $E_n = F_n = \ell_2^n$), we see that the diagonal operators $\text{diag}(R_k): F \rightarrow (\bigoplus \ell_1)_{\ell_1}$ and $\text{diag}(S_k): (\bigoplus \ell_1)_{\ell_1} \rightarrow \tilde{F}$ (where $\tilde{F} = (\bigoplus_{n=1}^\infty F)_{\ell_1}$) satisfy

$$\|\text{diag}(TJ_k) - \text{diag}(S_k) \text{diag}(R_k)\| = \sup \|TJ_k - S_kR_k\| \leq \varepsilon.$$

It follows that $\text{diag}(TJ_k) \in \overline{\mathcal{G}}_{\ell_1}(F, \tilde{F})$ because $(\bigoplus \ell_1)_{\ell_1}$ is isomorphic to ℓ_1 and ε is arbitrary, and so by (2.2) we conclude that $T \in \overline{\mathcal{G}}_{\ell_1}(F)$, as desired.

To show “(ii) \Rightarrow (iii)” suppose that $\sup\{n_\varepsilon(TJ_k) \mid k \in \mathbb{N}\} = \infty$ for some $\varepsilon > 0$. We construct inductively a strictly increasing sequence (k_j) of natural numbers such that, for each $j \in \mathbb{N}$, the following assertions hold:

- (a) $\text{colsupp}_{k_j}(T) \neq \emptyset$ and $\mu_{k_{j+1}}(T) \geq \mu_{k_j}(T)$.
- (b) Set $m_0 := 0$, $m_j := \mu_{k_j}(T)$, and $E_j := (\bigoplus_{i=m_{j-1}+1}^{m_j} \ell_2^i)_{\ell_1}$, and let $P_j: F \rightarrow E_j$ be the canonical projection. Then there are operators $U_j: \ell_2^{k_j} \rightarrow \ell_2^j$ and $V_j: E_j \rightarrow \ell_2^j$ such that the diagram

$$\begin{array}{ccc} \ell_2^j & \xrightarrow{I} & \ell_2^j \\ U_j \downarrow & & \uparrow V_j \\ \ell_2^{k_j} & \xrightarrow{J_{k_j}} F \xrightarrow{T} F \xrightarrow{P_j} & E_j \end{array}$$

is commutative, $\|U_j\| \leq 1/\varepsilon$, $\|V_j\| \leq 1$, and $\text{im } U_j \subseteq \bigcap_{i=1}^{m_j-1} \ker T_{i,k_j}$. (The latter condition is ignored for $j = 1$.)

We start the induction by choosing $k_1 \in \mathbb{N}$ such that $n_\varepsilon(TJ_{k_1}) \geq 1$. Then $\text{colsupp}_{k_1}(T) \neq \emptyset$ and $\|TJ_{k_1}\| > \varepsilon$. Take a unit vector $x \in \ell_2^{k_1}$ such that $\|TJ_{k_1}x\| > \varepsilon$, and define

$$U_1: \alpha \mapsto \frac{\alpha \sum_{i=1}^{k_1} x}{\|TJ_{k_1}x\|}, \quad \ell_2^1 \rightarrow \ell_2^{k_1}.$$

Further, take a functional $V_1: E_1 \rightarrow \ell_2^1$ of norm 1 such that $V_1(P_1TJ_{k_1}x) = \|P_1TJ_{k_1}x\|$. Then the diagram in (b) is commutative because $\|P_1TJ_{k_1}x\| = \|TJ_{k_1}x\|$.

Now let $j \geq 2$, and suppose that $k_1 < k_2 < \dots < k_{j-1}$ have been chosen in accordance with (a)–(b). Set $h := \sum_{i=1}^{m_{j-1}} i$, take $k_j > k_{j-1}$ such that $n_\varepsilon(TJ_{k_j}) \geq h + 2(j-1)$,

and set $H := \bigcap_{i=1}^{m_j-1} \ker T_{i,k_j} = \ker(\bigoplus_{i=1}^{m_j-1} Q_1) \circ T \circ J_{k_j}$. Since $\dim H \geq k_j - h$ it follows that $\dim H^\perp \leq h$ and hence $n_\varepsilon(TJ_{k_j}|_H) \geq 2(j-1)$ by Lemma 2.6(iii). In particular $TJ_{k_j}|_H \neq 0$, and (a) is satisfied. Further, we note that $n_\varepsilon(P_j TJ_{k_j}|_H) = n_\varepsilon(TJ_{k_j}|_H)$ because $Q_i TJ_{k_j}|_H = 0$ whenever $i \leq m_{j-1}$ or $i > m_j$. Lemma 2.6(ii) then implies that there are operators $U_j: \ell_2^j \rightarrow H \subseteq \ell_2^{k_j}$ and $V_j: E_j \rightarrow \ell_2^j$ such that (b) is satisfied. This completes the inductive construction.

We ‘glue’ the operators U_j ($j \in \mathbb{N}$) together in the following way to obtain an operator U on F . Given $x \in F$, define $y_i \in \ell_2^i$ by

$$y_i := \begin{cases} U_j Q_j x & \text{if } i = k_j \text{ for some } j \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \quad (i \in \mathbb{N}).$$

Then we have

$$\sum_{i=1}^{\infty} \|y_i\| = \sum_{j=1}^{\infty} \|U_j Q_j x\| \leq \frac{\|x\|}{\varepsilon} < \infty,$$

and so $Ux := (y_i)$ defines an operator U on F . Similarly, since

$$\sum_{j=1}^{\infty} \|V_j P_j x\| \leq \sum_{j=1}^{\infty} \|P_j x\| = \|x\|,$$

the assignment $Vx := (V_j P_j x)$ defines an operator V on F .

We *claim* that $VTU = I_F$. To this end, it suffices to check that

$$Q_i VTU J_j x = \begin{cases} x & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad (i, j \in \mathbb{N}, x \in \ell_2^j). \quad (2.3)$$

By definition, we have $Q_i VTU J_j x = V_i P_i T J_{k_j} U_j x$. For $i = j$, the diagram in (b), above, shows that this is x . For $i < j$, we have $U_j x \in \ker T_{h,k_j}$ ($1 \leq h \leq m_{j-1}$), and so

$$P_i T J_{k_j} U_j x = \sum_{h=m_{i-1}+1}^{m_i} J_h Q_h T J_{k_j} U_j x = \sum_{h=m_{i-1}+1}^{m_i} J_h T_{h,k_j} U_j x = 0.$$

For $i > j$, $P_i T J_{k_j} = \sum_{h=m_{i-1}+1}^{m_i} J_h T_{h,k_j} = 0$ because $T_{h,k_j} = 0$ whenever $h \geq m_j$. This completes the proof of the implication “(ii) \Rightarrow (iii)”.

Finally to deduce “(iii) \Rightarrow (i)” we need to observe that $\overline{\mathcal{G}_{\ell_1}}(F)$ is a proper ideal in $\mathcal{B}(F)$, and, thus, cannot contain T , since by (iii) I_F is contained in the ideal generated by T . First note that by Lemma 2.8, if $I_F \in \overline{\mathcal{G}_{\ell_1}}(F)$, then $I_F \in \mathcal{G}_{\ell_1}(F)$, and hence F would be isomorphic to ℓ_1 . It is known, however, that F is not isomorphic to ℓ_1 , although this is by no means obvious. One may for example use that ℓ_1 has a unique unconditional basis up to equivalence (a fact that essentially relies on Khintchine’s inequality), whereas it is easy to see that F does not have this property. \square

Our main theorem on the classification of the closed ideals of $\mathcal{B}(F)$ can be deduced.

2.10 Theorem. Set $F := (\bigoplus \ell_2^n)_{\ell_1}$. The lattice of closed ideals in $\mathcal{B}(F)$ is given by

$$\{0\} \subsetneq \mathcal{K}(F) \subsetneq \overline{\mathcal{G}}_{\ell_1}(F) \subsetneq \mathcal{B}(F). \quad (2.4)$$

Proof. It is clear that $\{0\} \subsetneq \mathcal{K}(F) \subsetneq \overline{\mathcal{G}}_{\ell_1}(F)$ (F contains ℓ_1 as a complemented subspace, the projection onto which is an example of a non-compact operator in $\overline{\mathcal{G}}_{\ell_1}(F)$). By (ii) \iff (i) in Theorem 2.9 the identity on F cannot be in $\overline{\mathcal{G}}_{\ell_1}(F)$, thus $\overline{\mathcal{G}}_{\ell_1}(F)$ is a proper subideal of $\mathcal{B}(F)$.

We now show that the ideals in (2.4) are the only closed ideals of $\mathcal{B}(F)$. Standard basis arguments show that the identity on ℓ_1 factors through any non-compact operator in $\mathcal{B}(F)$ (see for example [7, §3]). It follows that for each non-zero, closed ideal \mathcal{J} in $\mathcal{B}(F)$, either $\mathcal{J} = \mathcal{K}(F)$ or $\overline{\mathcal{G}}_{\ell_1}(F) \subseteq \mathcal{J}$.

Suppose that \mathcal{J} is a closed ideal in $\mathcal{B}(F)$ properly containing $\overline{\mathcal{G}}_{\ell_1}(F)$. Take $T \in \mathcal{J} \setminus \overline{\mathcal{G}}_{\ell_1}(F)$, and take $\tilde{T} \in \mathcal{B}(F)$ with finite columns such that $T - \tilde{T}$ is compact (cf. [7, Lemma 2.7(i)]). Then \tilde{T} is also in $\mathcal{J} \setminus \overline{\mathcal{G}}_{\ell_1}(F)$. By Theorem 2.9 we conclude that $I_F = V\tilde{T}U$ for some operators U and V on F . It follows that $\mathcal{J} = \mathcal{B}(F)$, as required. \square

In [1, §8] Bourgain, Casazza, Lindenstrauss, and Tzafriri prove that every infinite-dimensional, complemented subspace of the Banach space $F := (\bigoplus \ell_2^n)_{\ell_1}$ is isomorphic to either F or ℓ_1 . Here we present a new proof of this fact using only the ideal structure of $\mathcal{B}(F)$. More precisely, we shall deduce it from the dichotomy in Theorem 2.10 for operators in $\mathcal{B}(F)$ with finite columns.

2.11 Remark. In [7, §6] a new proof is presented for the corresponding result of Bourgain, Casazza, Lindenstrauss, and Tzafriri for the space $E := (\bigoplus \ell_2^n)_{c_0}$, which says that every infinite-dimensional, complemented subspace of E is isomorphic to either E or c_0 . This new proof in [7] relies on a result of Casazza, Kottman and Lin [3] that implies that E is primary. The result of [3], however, does not show that F is primary, and so the argument in [7] cannot be used here. The proof we present below uses only the classification result, Theorem 2.10, and it also works for the space E .

We start with an easy strengthening of part (ii) of Theorem 2.10.

2.12 Proposition. Let T be an operator on F . If $T \notin \overline{\mathcal{G}}_{\ell_1}(F)$ then there exist operators A and B on F such that $I_F = ATB$.

Proof. Let K be a compact operator on F such that $T - K$ has finite columns. Note that by the ideal property we have $T - K \notin \overline{\mathcal{G}}_{\ell_1}(F)$. By Theorem 2.10 there are operators U and V on F such that $I_F = U(T - K)V$. Thus UTV is a compact perturbation of the identity, and hence it is a Fredholm operator. It follows that for some $W \in \mathcal{B}(F)$ the operator $WUTV$ is a cofinite-rank projection. Since E is isomorphic to its finite-codimensional subspaces the result follows. \square

2.13 Theorem. (Bourgain, Casazza, Lindenstrauss, Tzafriri [1]) Every infinite-dimensional, complemented subspace of $F = (\bigoplus \ell_2^n)_{\ell_1}$ is isomorphic to either F or ℓ_1 .

Proof. Let Y be an infinite-dimensional, complemented subspace of F , and let $P \in \mathcal{B}(F)$ be an idempotent operator with image Y . If $P \in \overline{\mathcal{G}}_{\ell_1}(F)$, then by Lemma 2.8 we have $P \in \mathcal{G}_{\ell_1}(F)$, and hence Y is isomorphic to ℓ_1 . If $P \notin \overline{\mathcal{G}}_{\ell_1}(F)$, then by Proposition 2.12 the identity on F factors through P , i.e., F is isomorphic to a complemented subspace of Y . We can thus write $F \sim Y \oplus V$ and $Y \sim F \oplus W$ for suitable Banach spaces V and W . We now use Pelczynski's decomposition method to show that Y is isomorphic to F .

$$\begin{aligned}
F &\sim Y \oplus V \\
&\sim F \oplus W \oplus V \\
&\sim (F \oplus F \oplus \dots)_{\ell_1} \oplus W \oplus V \\
&\sim (Y \oplus V \oplus Y \oplus V \oplus \dots)_{\ell_1} \oplus W \oplus V \\
&\sim (Y \oplus V \oplus Y \oplus V \oplus \dots)_{\ell_1} \oplus W \\
&\sim F \oplus W \sim Y,
\end{aligned}$$

where we also used the fact that F is isomorphic to $(F \oplus F \oplus \dots)_{\ell_1}$. □

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References

- [1] J. Bourgain, P. G. Casazza, J. Lindenstrauss, and L. Tzafriri, *Banach spaces with a unique unconditional basis, up to permutation*, Mem. American Math. Soc. **322**, 1985.
- [2] J. W. Calkin, Two-sided ideals and congruences in the ring of bounded operators in Hilbert space, *Annals of Math.* **42** (1941), 839–873.
- [3] P. G. Casazza, C. A. Kottman, Bor Luh Lin, *On some classes of primary Banach spaces*, Canad. J. Math. **29** (1977), no. 4, 856–873.
- [4] J. Diestel, A survey of results related to the Dunford–Pettis property, in *Proceedings of the conference on integration, topology, and geometry in linear spaces* (ed. W. H. Graves), Contemp. Math. **2**, American Math. Soc., Providence, R. I., 1980.
- [5] I. C. Gohberg, A. S. Markus, and I. A. Feldman, Normally solvable operators and ideals associated with them, *American Math. Soc. Translat.* **61** (1967), 63–84, Russian original in *Bul. Akad. Štiințe RSS Moldoven* **10** (76) (1960), 51–70.

- [6] B. Gramsch, Eine Idealstruktur Banachscher Operatoralgebren, *J. Reine Angew. Math.* **225** (1967), 97–115.
- [7] N. J. Laustsen, R. J. Loy, and C. J. Read, The lattice of closed ideals in the Banach algebra of operators on certain Banach spaces, *J. Functional Anal.*, to appear.
- [8] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Vol. I, *Ergeb. Math. Grenzgeb.* **92**, Springer-Verlag, 1977.
- [9] E. Luft, The two-sided closed ideals of the algebra of bounded linear operators of a Hilbert space, *Czechoslovak Math. J.* **18** (1968), 595–605.
- [10] A. Pietsch, *Operator ideals*, North-Holland, 1980.

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