ON THE CLOSED SUBIDEALS OF $L(\ell_p \oplus \ell_q)$

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Abstract. In this paper we first review the known results about the closed subideals of the space of bounded operator on $\ell_p \oplus \ell_q$, $1 < p < q < \infty$, and construct several new ones.

1. Introduction

For very few Banach spaces X all the closed subideals of $L(X)$, the algebra of all bounded and linear operators on X, are determined. In 1941 Calkin [5] showed that the only proper, non-trivial and closed ideal of $L(\ell_2)$ is the ideal of compact operators. The same was shown to be true for ℓ_p ($1 \le p < \infty$) and c_0 in [11]. Until very recently it was open if there are any other infinite dimensional Banach spaces X, for which the compact operators are the only proper, non-trivial and closed subideal of $L(X)$. We call such spaces *simple*. Then Argyros and Haydon [3] established the existence of Banach spaces with a basis on which all operators are a compact perturbation of a scalar multiple of the identity. Trivially, such spaces are simple. But it is not known whether or not there any other simple spaces admitting an unconditional basis (and thus having a rich structure of operators on them).

The structure of the closed ideals of operators on non separable Hilbert spaces was independently obtained by Gramsch [12] and Luft [17]. Recently Daws [6] extended their results to non separable ℓ_p-spaces, $1 \le p < \infty$, and non separable c_0-spaces.

Beyond these spaces the complete structure of closed ideals in $L(X)$ was described in [13] for $X = \big(\bigoplus_{n=1}^\infty \ell_2(n)\big)_{c_0}$ and in [15] for $X = \big(\bigoplus_{n=1}^\infty \ell_2(n)\big)_{\ell_1}$. In both cases, there are exactly two nested proper non-zero closed ideals, namely the compacts and the closure of all operators factoring through c_0, or ℓ_1, respectively. Apart from those mentioned above, there are no other separable Banach spaces X for which the structure of the closed ideals in $L(X)$ is completely known. It is still open whether or not the closed subideals of the operators on the spaces $(\oplus_{n=1}^\infty \ell_1(n))_{c_0}$ and $(\oplus_{n=1}^\infty \ell_\infty(n))_{\ell_1}$ admit the same sublattice structure (for partial results see [14]). An interesting space for studying the closed subideals of its bounded linear operators is the space X introduced in [22]. This space is *complementably minimal* [1], which means that every infinite dimensional closed subspace of X contains a further subspace which is complemented in X and isomorphic to X. This implies that the strictly singular operators (see the definition at the end of this section) is the only maximal proper closed subideal of $L(X)$. As shown in [2], X admits strictly singular but not compact operators, and it

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is conjectured that $L(X)$ contains infinitely many closed subideals, all of which have to lie between the ideal of compact operators, and the ideal of strictly singular operators.

Although studied in several papers (cf. [18],[20] and [21]) the structure of the closed subideals of $L(\ell_p \oplus \ell_q)$, $1 < p < q < \infty$ remains a mystery. It is not even known whether or not $L(\ell_p \oplus \ell_q)$, contains infinitely many subideals. There were several results proved in the 1970's concerning various special ideals or special cases of p and q. We refer the reader to the book by Pietsch [20, Chapter 5] for details. In particular, [20, Theorem 5.3.2] asserts that $L(\ell_p \oplus \ell_q)$, with $1 \le p < q$, has exactly two proper maximal ideals (namely, the ideal of operators which factor through ℓ_p and the ideals of operators which factor through ℓ_q), and establishes a one-to-one correspondence between the non-maximal proper subideals of $L(\ell_p \oplus \ell_q)$ and the closed ideals in $L(\ell_p, \ell_q)$. By proving that the formal identity $I(p,q) : \ell_p \to \ell_q$ is *finitely strictly singular* (see the definition at the end of this section) and establishing the existence of an operator $T : \ell_p \to \ell_q$ which is not finitely strictly singular Mliman [18] concluded that $L(\ell_p, \ell_q)$ contains at least two non trivial, proper and closed subideals. In [21] the study of the structure of the closed subideals of $L(\ell_p, \ell_q)$ was continued, and, among other results, it was discovered that the lattice of subideals of $L(\ell_p, \ell_q)$ is not linearly ordered, and contains at least 4 nontrivial, proper and closed subideals if $1 < p < 2 < q < \infty$. In this paper we increase this number to 7.

In Section 2 we will recall the known results on the closed subideals of $L(\ell_p \oplus \ell_q)$ and $L(\ell_p, \ell_q)$, and sketch the proof of several of them. In Section 3 we will formulate and prove our main result (see Theorem 3.1).

Let us first recall some necessary notation.

If X and Y are Banach spaces, $L(X,Y)$ denotes the space of bounded linear operators $T : X \to Y$, and if $X = Y$ we write $L(X)$ instead of $L(X,X)$. A linear subspace $\mathcal{J} \subset L(X,Y)$, is called a *subideal of* $L(X,Y)$, if for all $A \in L(Y)$, $B \in L(X)$, and $T \in \mathcal{J}$ also $A \circ T \circ B \in \mathcal{J}$. A *closed subideal of* $L(X,Y)$ is a subideal which is closed in the operator norm. We say that a subideal $\mathcal{J} \subset L(X,Y)$ is *non trivial* if it is not the *zero ideal* $\{0\}$ and proper if it is not all of $L(X,Y)$.

The following is a list of some important closed subideals of $L(X,Y)$.

$\mathcal{FD}(X,Y)$ is the closure of the ideal of operators with finite dimensional rank. Note that any nontrivial closed subideal $\mathcal{J}$ in $L(X,Y)$ contains all of $\mathcal{FD}(X,Y)$. This follows from the fact that $\mathcal{J}$ is closed under taking sums, under multiplication by elements of $L(X)$ from the right, under multiplication from the left by elements of $L(Y)$, and that it must contain a non zero operator (and thus a rank 1 operator). Thus, for all infinite dimensional Banach spaces X and Y the ideal $\mathcal{FD}(X,Y)$ is the minimal nontrivial closed subideal of $L(X,Y)$.

$\mathcal{K}(X,Y)$ denotes the ideal of compact operators . All the spaces we consider are spaces with a basis. Thus, these spaces have the approximation property, which means that $\mathcal{FD}(X,Y) = \mathcal{K}(X,Y)$.

$\mathcal{S}t\mathcal{S}i(X,Y)$ is the closed ideal of operators $T : X \to Y$ which are *strictly singular*, i.e. on no infinite dimensional subspace Z of X is the restriction of T onto Z an isomorphism.

$\mathcal{FSS}$ is the closed ideal of *finitely strictly singular operators.* A linear bounded operator $T : X \to Y$, is called *finitely strictly singular* if for all $\varepsilon > 0$ there is an $n = n_\varepsilon \in \mathbb{N}$ so that in any n-dimensional subspace E of X, there is an $x \in E$, with $\|x\| = 1$, so that $\|T(x)\| \le \varepsilon$.

If W and Z are Banach spaces and $S : W \to Z$ a bounded linear operator, we denote by $\mathcal{J}^S(X,Y)$ the closure of the ideal generated by all operators $T \in L(X,Y)$, which factor through S, thus $T = A \circ S \circ B$, with $A \in L(Z,Y)$ and $B \in L(X,W)$. In general the set $\{A \circ S \circ B, A \in L(Z,Y)$ and $B \in L(X,W)\}$ is not closed under addition and therefore not an ideal. But if the operator

$$S \oplus S : W \oplus W \to Z \oplus Z, \qquad (w_1, w_2) \mapsto (S(w_1), S(w_2)),$$

factors through S, then $\{A \circ S \circ B, A \in L(Z,Y)$ and $B \in L(X,W)\}$ is an ideal and we conclude in that case that

$$\mathcal{J}^S(X,Y) = \overline{\{A \circ S \circ B : A \in L(Z,Y) \text{ and } B \in L(X,W)\}}. \tag{1}$$

Let $I(p,q) : \ell_p \to \ell_q$ be the formal inclusion (using that ℓ_p is a subset of ℓ_q), for $1 \le p < q \le \infty$. It is easily seen that $I(p,q) \oplus I(p,q)$ factors through $I(p,q)$ and we conclude that $\mathcal{J}^{I(p,q)}(X,Y) = \overline{\{A \circ I(p,q) \circ B : A \in L(\ell_q, Y) \text{ and } B \in L(X, \ell_p)\}}$.

If I_Z is the identity on some Banach space Z we write $\mathcal{J}^Z$ instead of $\mathcal{J}^{I_Z}$, and we note that if Z is isomorphic to $Z \oplus Z$ it follows that

$$\mathcal{J}^Z(X,Y) = \overline{\{A \circ S \circ B : A \in L(Z,Y) \text{ and } B \in L(X,Z)\}}. \tag{2}$$

If $X = Y$ we will write $\mathcal{K}(X)$, $\mathcal{FSS}(X)$ etc. instead of $\mathcal{K}(X,X)$, $\mathcal{FSS}(X,X)$ etc.

For $1 \le p < \infty$, we denote the unit vector basis of $\ell_p = \ell_p(\mathbb{N})$ by $(e_{(p,j)} : j \in \mathbb{N})$ (if $p = \infty$ we consider c_0 instead of ℓ_∞). The conjugate of p is denoted by p', i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. For $n \in \mathbb{N}$ we denote the n-dimensional ℓ_p space by $\ell_p(n)$ and its unit vector basis by $(e_{(p,n,j)} : j \in \mathbb{N})$. The usual norm on ℓ_p or $\ell_p(n)$, $n \in \mathbb{N}$ is denoted by $\|\cdot\|_p$. If X_n is a Banach space for $n \in \mathbb{N}$, the *ℓ_p-sum of* X_n, $n \in \mathbb{N}$, is the space of all sequences $(x_n : n \in \mathbb{N})$, with $x_n \in X_n$, for $n \in \mathbb{N}$, and

$$\|(x_n)_{n \in \mathbb{N}}\|_p = \Big(\sum_{n \in \mathbb{N}} \|x_n\|^p\Big)^{1/p} < \infty, \text{ if } p < \infty, \text{ and}$$

We denote the ℓ_p sum of (X_n) by $(\oplus_{n=1}^\infty X_n)_p$. If $p = \infty$ we denote by $(\oplus_{n=1}^\infty X_n)_\infty$ the *c_0-sum,* the space of all sequences (x_n), with $x_n \in X_n$, for $n \in \mathbb{N}$, for which $\lim_{n \to \infty} \|x_n\| = 0$.

The sphere and the unit ball of a Banach space are denoted by S_X and B_X, respectively. For simplicity all our Banach spaces are defined over the real field $\mathbb{R}$. It is easy to see how our results can be extended to Banach spaces over the complex field $\mathbb{C}$.

2. REVIEW OF THE KNOWN RESULTS ON THE CLOSED SUBIDEALS OF $L(\ell_p \oplus \ell_q)$ AND $L(\ell_p, \ell_q)$

We will now review the known results on the lattice structure of subideals of $L(\ell_p \oplus \ell_q)$. We will assume from now on that $1 < p < q < \infty$ and later that $1 < p < 2 < q < \infty$.

Every operator $T = \ell_p \oplus \ell_q \to \ell_p \oplus \ell_q$, consists of four operators $T_{(1,1)} \in L(\ell_p)$, $T_{(1,2)} \in L(\ell_q, \ell_p)$ and $T_{(2,1)} \in L(\ell_p, \ell_q)$, and $T_{(2,2)} \in L(\ell_p, \ell_p)$, and acts as a 2 by 2 matrix on the elements of $\ell_p \oplus \ell_q$

$$T = \begin{pmatrix} T_{(1,1)} & T_{(1,2)} \\ T_{(2,1)} & T_{(2,2)}T \end{pmatrix} : \ell_p \oplus \ell_q \to \ell_p \oplus \ell_q, \quad (x, y) \mapsto \big(T_{(1,1)}(x) + T_{(1,2)}(y), T_{(2,1)}(x) + T_{(2,2)}(y)\big).$$

By the above cited result from [11], the operators $T_{(1,1)}$ and $T_{(2,2)}$ are either compact or the identity on ℓ_p, respectively ℓ_q, factors through them. By Pitt's Theorem (c.f. [9, Proposition 6.25]), $T_{(1,2)}$ is compact, and since every infinite dimensional subspace of ℓ_p contains a subspace isomorphic to ℓ_p, and since ℓ_p and ℓ_q are incomparable, we conclude that $T_{(2,1)}$ must be strictly singular. So, if $\mathcal{J}$ is a closed subideal of $L(\ell_p \oplus \ell_q)$ which contains an operator T for which $T_{(1,1)}$ and $T_{(2,2)}$ are not compact, we conclude that the identity on $\ell_p \oplus \ell_q$ factors through T and thus $\mathcal{J} = L(\ell_p \oplus \ell_q)$. If $\mathcal{J}$ contains an operator for which $T_{(1,1)}$ is not compact, but for all elements $U \in \mathcal{J}$, $U_{(2,2)}$ is compact, then the identity on ℓ_p factors through T, but not the identity on ℓ_q, and we therefore deduce that J must be the closure of the operators factoring through ℓ_p, which must therfore be a maximal proper subideal of $L(\ell_p \oplus \ell_q)$ (for more details see [20, Theorem 5.3.2]). Similarly we conclude that the closure of all operators factoring through ℓ_q is a maximal proper subideal of $L(\ell_p \oplus \ell_q)$.

For all other closed proper subideals $\mathcal{J} \subset L(\ell_p \oplus \ell_q)$, and all $T \in \mathcal{J}$ it therefore follows that $T_{(1,1)}$, $T_{(1,2)}$ and $T_{(2,2)}$ are compact, and can therefore be approximated by finite rank operators which factor through ℓ_p as well as ℓ_q. Of course $T_{(2,1)}$ also factors through ℓ_p as well as ℓ_q, and we deduce that all other closed ideals are subideals of $\mathcal{J}^{\ell_p}(\ell_p \oplus \ell_q) \cap \mathcal{J}^{\ell_q}(\ell_p \oplus \ell_q)$, and thus not maximal proper closed ideals.

Assume now that $\mathcal{J} \subset \mathcal{J}^{\ell_p}(\ell_p \oplus \ell_q) \cap \mathcal{J}^{\ell_p}(\ell_p \oplus \ell_q)$ is a closed ideal in $L(\ell_p \oplus \ell_q)$ An easy computation yields that $\tilde{\mathcal{J}} := \{T_{(2,1)} : T \in \mathcal{J}\}$ is a closed subideal of $L(\ell_p, \ell_q)$, and that for two different two ideals $\mathcal{J}_1, \mathcal{J}_2 \subset \mathcal{J}^{\ell_p}(\ell_p \oplus \ell_q) \cap \mathcal{J}^{\ell_p}(\ell_p \oplus \ell_q)$ the ideals $\tilde{\mathcal{J}}_1$ and $\tilde{\mathcal{J}}_2$ are different. Conversely if $\mathcal{J}$ is a closed subideal of $L(\ell_p, \ell_q)$ then

$$\mathcal{J}' = \left\{ \begin{pmatrix} T_{(1,1)} & T_{(1,2)} \\ T_{(2,1)} & T_{(2,2)} \end{pmatrix} : T_{(2,10} \in \mathcal{J} \text{ and } T_{(1,1)} \in \mathcal{K}(\ell_p),\ T_{(1,2)} \in \mathcal{K}(\ell_q, \ell_p),\ \text{and } T_{(2,2)} \in \mathcal{K}(\ell_q) \right\}$$

is a closed subideal of $L(\ell_p \oplus \ell_q)$ and for two different closed subideals $J_1, J_2 \subset L(\ell_p, \ell_q)$, $\mathcal{J}_1'$ and $\mathcal{J}_2'$ are different. Thus there is a bijection between the set of all closed subideals of $L(\ell_p, \ell_q)$ and the non maximal closed subideals of $L(\ell_p, \ell_q)$, which preserves the lattice structure with respect to inclusions.

We are therefore interested in the closed subideals of $L(\ell_p, \ell_q)$. Instead of writing $\mathcal{K}(\ell_p, \ell_q)$, $\mathcal{FSS}(\ell_p, \ell_q)$, or $\mathcal{J}^S(X, Y)$ etc. we will from now on simply write $\mathcal{K}$, $\mathcal{FSS}$ or $\mathcal{J}^S$ etc.

The following diagram summarizes the results established in [18] and [21], under the assumption that $1 < p < 2 < q < \infty$.

$$\begin{tikzcd}
 & & & \boxed{\mathcal{FSS}} \arrow[dr, Rightarrow] \arrow[dd, phantom, "\nparallel"] & & \\
\boxed{\mathcal{K}} \arrow[r, Rightarrow] & \boxed{\mathcal{J}}^{I(p,q)} \arrow[r, dotted] & \boxed{\mathcal{FSS}\cap\mathcal{J}^{\ell_2}} \arrow[ur, dashed] \arrow[dr] & & \boxed{\mathcal{FSS}\vee\mathcal{J}^{\ell_2}} \arrow[r, dotted] & \boxed{L(\ell_p,\ell_q)} \\
 & & & \boxed{\mathcal{J}^{\ell_2}} \arrow[ur, dashed] & &
\end{tikzcd}$$

Here arrows stand for inclusions. A solid arrow ($\Rightarrow$ or $\rightarrow$) between two ideals means that there are no other ideals sitting properly between the two, while a double arrow coming out of an ideal indicates the only immediate successor. A hyphenated arrow ($-->$) indicates a proper inclusion, while a dotted one indicates that we do not know whether or not the inclusion is proper. In particular, the closed ideals in $L(\ell_p,\ell_q)$ are not totally ordered.

Let us explain the diagram "from the left to the right" (for a more detailed explanation we refer the reader to [21]):

If $T:\ell_p\to\ell_q$ is not compact, then there is a normalized block sequence (x_n) in ℓ_p whose image $(y_n)=(T(x_n)$ is equivalent to $(e_{(q,j)}:j\in\mathbb{N})$ (the unit vector basis in ℓ_q) and so that $\operatorname{span}(y_n:n\in\mathbb{N})$ is complemented in ℓ_p. It follows that $I(p,q)$ factors through T, and that therefore $\mathcal{J}^{I(p,q)}$ is the only successor of $\mathcal{K}$.

It is clear that $\mathcal{J}^{I(p,q)}\subset\mathcal{J}^{\ell_2}$ (recall that we assume that $p<2<q$). The fact that $\mathcal{J}^{I(p,q)}\subset\mathcal{FSS}$ follows from the following result in [18] (see also [21, Proposition 3.3]).

Proposition 2.1. *For any choices of* $1\le p<q\le\infty$ *is the formal identity* $I(p,q)$ *is a finitely strictly singular operator.*

The way to verify Proposition 2.1 is to show first (see [18] or [21, Lemma 3.4]) by induction on $n\in\mathbb{N}$, that in every n-dimensional subspace E of c_0 there is $x\in E$ which attains its sup-norm on at least n coordinates. In order to see then, that $I(p,q)$ is finitely strictly singular, let $\varepsilon>0$ and pick $n\in\mathbb{N}$ with $n^{\frac{1}{q}-\frac{1}{p}}<\varepsilon$. If E is any subspace of ℓ_p of dimension n we can find $x\in E$, $\|x\|=1$, so that $\|x\|_\infty\le n^{-1/p}$ (since the maximum is attained on at least n coordinates), and thus $\|x\|_q^q=\sum_{i=1}^n|x(i)|^{q-p}|x(i)|^p\le\|x\|_\infty\le n^{q-p}\|x\|_p^p\le n^{-(q-p)/p}$ and thus $\|x\|_q\le n^{-(q-p)/pq}\le\varepsilon$. We therefore established that $\mathcal{J}^{I(p,q)}\subset\mathcal{FSS}\cap\mathcal{J}^{\ell_2}$. In Section 2 we will show that this inclusion is strict.

In order to show that $\mathcal{FSS}\cap\mathcal{J}^{\ell_2}$ is not all of $L(\ell_p,\ell_q)$ Milman [18] used the fact that ℓ_p (and ℓ_q)is isomorphic the ℓ_p-sum (respectively the ℓ_q sum) of $\ell_2(n)$, $n\in\mathbb{N}$ (see [16, page 73]). Letting $U:\ell_p\to(\oplus_{n\in\mathbb{N}}\ell_2(n))_p$ and $V:\ell_q\to(\oplus_{n\in\mathbb{N}}\ell_2(n))_q$ be isomorphisms and letting $I'(p,q)$ be the formal identity

$$I'(p,q):(\oplus_{n\in\mathbb{N}}\ell_2(n))_p\to(\oplus_{n\in\mathbb{N}}\ell_2(n))_q,\qquad (x_n)\mapsto(x_n),$$

we define $T(p,q)=V\circ I'(p,q)\circ U$. $T(p,q)$ depends on the choice of the isomorphisms U and V, nevertheless it is easy to see that for any other isomorphisms $\tilde{U}:\ell_p\to(\oplus_{n\in\mathbb{N}}\ell_2(n))_p$ and $\tilde{V}:\ell_q\to(\oplus_{n\in\mathbb{N}}\ell_2(n))_q$, the operator $\tilde{T}(p,q)=\tilde{V}\circ I'(p,q)\circ\tilde{U}$, factors through $T(p,q)$ and vice versa, and thus $\mathcal{J}^{T(p,q)}=\mathcal{J}^{\tilde{T}(p,q)}$. Clearly $T(p,q)\notin\mathcal{FSS}$, and thus $\mathcal{FSS}$ is a proper closed subideal of $L(\ell_p,\ell_q)$.

It is clear that $\mathcal{J}^{T(p,q)} \subset \mathcal{J}^{\ell_2}$. Conversely, Theorem 4.7 in [21] shows that every operator $S : \ell_p \to \ell_q$, which factors through ℓ_2, belongs to $\mathcal{J}^{T(p,q)}$, thus we deduce that $\mathcal{J}^{T(p,q)} = \mathcal{J}^{\ell_2}$. Moreover, if $S \in L(\ell_p, \ell_q)$ is not in $\mathcal{FSS}$, it follows from Khintchine's theorem (for more detail see Theorem 3.2 in Section 3 and the remarks thereafter) that for some $c > 0$ there are c-complemented subspaces $F_n \subset \ell_p$, which are c-isomorphic to $\ell_2(n)$, for $n \in \mathbb{N}$, on which S is a c-isomorphism. After perturbing S we can find a sequence $(k_n) \subset \mathbb{N}$, so that if we write ℓ_p as an ℓ_p-sum of $\ell_p(k_n)$ and ℓ_q as the ℓ_q-sum of $\ell_q(k_n)$, we can assume that $F_n \subset \ell_p(k_n) \subset \ell_p$ and $S(F_n) \subset \ell_q(k_n) \subset \ell_q$. From this (see [21, Theorem 4.13]) it follows that $T(p,q)$ factors through S. We deduce therefore that the ideal $\mathcal{J}^{\ell_2} \vee \mathcal{FSS} = \mathcal{J}^{T(p,q)} \vee \mathcal{FSS}$ (the closed ideal generated by the elements of $\mathcal{FSS}$ and $\mathcal{J}^{\ell_2}$) is the only successor of $\mathcal{FSS}$.

Finally we need to construct an operator $U : \ell_p \to \ell_q$ which is in $\mathcal{FSS}$ but cannot be approximated by operators which factor through ℓ_2. This will show that $\mathcal{FSS}$ and $\mathcal{J}^{\ell_2}$ are incomparable, they both strictly contain $\mathcal{FSS} \cap \mathcal{J}^{\ell_2}$ and are properly contained in $\mathcal{J}^{\ell_2} \vee \mathcal{FSS}$.

To do that we write ℓ_p as ℓ_p sum of $\ell_p(2^n)$, $n \in \mathbb{N}$, and ℓ_q as ℓ_q-sum of $\ell_q(2^n)$, $n \in \mathbb{N}$. For $n \in \mathbb{N} \cup \{0\}$ let H_n be the n-th Hadamard matrix. This is an 2^n by 2^n matrix with entries which are either 1 or -1, and can be defined by induction as follows; $H_0 = (1)$, and assuming that H_n has been defined one puts $H_{n+1} = \begin{pmatrix} H_n & H_n \\ H_n & -H_n \end{pmatrix}$.

It is easy to see that H_n as operator from $\ell_1(2^n) \to \ell_\infty(2^n)$ is of norm 1, and that $2^{-n/2} H_n$ is a unitary matrix (i.e., an isometry on $\ell_2(2^n)$). It follows therefore from the Riesz Thorin Interpolation Theorem (c.f. [4]) that $U_n = 2^{-n\frac{1}{\min(p',q)}} H_n$ is of norm at most 1 as an operator in $L(\ell_p(2^n), \ell_q(2^n))$.

We define

$$U : \ell_p = \big(\oplus_{n=1}^\infty \ell_p(2^n) \big)_p \to \big(\oplus_{n=1}^\infty \ell_p(2^n) \big)_q, \qquad (x_n) \mapsto (U_n(x_n)).$$

The fact that U can not be approximated by operators which factor though ℓ_2 can be obtained from the following Corollary of Theorem 9.13 in [8] (see also [21, Theorem]).

Proposition 2.2. cf. [21, Corollary] *Let $m \in \mathbb{N}$, $C > 1$, and $r > 1$, and assume that V is an invertible m by m matrix. Let $\delta = \|V^{-1}\|_{L(\ell'_r, \ell_{r'})}$. Then $\|B\|_{L(\ell_p,\ell_r)} \cdot \|A\|_{L(\ell_r,\ell_q)} \geq \delta^{-1}$ for any factorization $V = AB$. Moreover, if $\widetilde{V}$ is another m by m matrix with*

$$\|\widetilde{V} - V\|_{L(\ell_p,\ell_q)} \leq \big(2 \max_{1\leq i\leq m} \|V^{-1} e_i\|_p\big)^{-1}, \tag{3}$$

then it follows that for any factorization $\widetilde{V} = AB$ we have $\|B\|_{L(\ell_p,\ell_r)} \cdot \|A\|_{L(\ell_r,\ell_q)} \geq (2\delta)^{-1}$.

If $q \neq p'$ then it is easy to see that U is finitely strictly singular. Indeed if $p' < q$, it follows that $U_n = 2^{-n/p'} H_n$, and we deduce again form the Riesz Thorin Interpolation Theorem that U_n is as operator between $\ell_p(2^n)$ and $\ell_{p'}(2^n)$ of norm not larger than 1, and thus $U \in L(\ell_p, \ell_{p'})$. But this implies that U (as element in $L(\ell_p, \ell_q)$) factors through $I(p', q)$, which is finitely strictly singular by Proposition 2.2. A similar argument shows that if $p' > q$, and thus $p < q'$, then U factors through $I(p, q')$.

The hard case is the case $q = p' \neq 2$, in which the previous factorization argument does not work. In this case it is better to see $\ell_p(n)$ as the space $L_p(n)$, the space of all p- integrable functions on $\{1, 2 \dots n\}$ with the normalized counting measure (i.e. $\|x\|_{L_p} = \frac{1}{n^{/1p}}\|x\|_p)$). Using interpolation between Schatten p-classes one can prove the following result

Theorem 2.3. [21, Theorem 6.5] *Suppose that* $T\colon L_p(N) \to \ell_{p'}(N)$*. Let* E *be a* k*-dimensional subspace of* $L_p(N)$*, and* C_1*,* C_2*, and* C_3 *be positive constants such that*

(1) $\|T\|_{L(L_2(N),\ell_2(N))} \leq 1$ *and* $\|T\|_{L(L_1(N),\ell_\infty(N)))} \leq 1$*;*
(2) E *is* C_1*-isomorphic to* ℓ_2^k*;*
(3) $F = T(E)$ *is* C_2*-complemented in* $\ell_{p'}^N$*; and*
(4) $T_{|E}$ *is invertible and* $\left\|(T_{|E})^{-1}\right\| \leq C_3$*.*

Then $k \leq \left(C_1^3 C_2 C_3^2 K_G^2\right)^{p'}$*.*

Now, if $q = p'$, then we apply for $n \in \mathbb{N}$ Theorem 2.3 to $N = 2^n$ and $T_n = \frac{1}{n^{1/p}} U_n = \frac{1}{n} H_n$ (note T_n satisfies (1) of Theorem 2.3). If U where not finitely strictly singular, we could find constants C_1, C_2 and C_3 and for any $k \in \mathbb{N}$ we would could find $n = n_k \in \mathbb{N}$ large enough so that (2) and (3) of Theorem 2.3 are satisfied (using again Theorem 3.2 in Section 2) for $T = T_n$. But this contradicts the conclusion of Theorem 2.3.

3. Two new closed ideals of $L(\ell_p, \ell_q)$

We now state our main result, which exhibits two new closed subideals of $L(\ell_p, \ell_q)$, and shows that $\mathcal{J}^{I(p,q)} \subsetneq \mathcal{FSS} \cap \mathcal{J}^{\ell_2}$ and increases the count of known closed proper and non trivial subideals of $L(\ell_p, \ell_q)$ to 7.

Theorem 3.1. *Assume that* $1 < p < 2 < q < \infty$*. Then the two ideals* $\mathcal{J}^{I(p,2)}$ *and* $\mathcal{J}^{I(2,q)}$ *are two incomparable closed subideals of* $\mathcal{FSS} \cap \mathcal{J}^{\ell_2}$*.*

We assume from now on that $1 < p < 2 < q < \infty$. It is clear that $\mathcal{J}^{I(p,q)} \subset \mathcal{J}^{I(p,2)}$ and that by Proposition 2.1 $\mathcal{J}^{I(p,2)} \subset \mathcal{FSS} \cap \mathcal{J}_2^{\ell}$ and similarly $\mathcal{J}^{I(p,q)} \subset \mathcal{J}^{I(2,q)} \subset \mathcal{FSS} \cap \mathcal{J}^{\ell_2}$. We can therefore extend the diagram of Section 2 to the following diagram.

$$\begin{tikzcd}
 & & \boxed{\mathcal{J}^{I(p,2)}} \arrow[dr, dashed] & & \boxed{\mathcal{FSS}} \arrow[dr, Rightarrow] & & \\
\boxed{\mathcal{K}} \arrow[r, Rightarrow] & \boxed{\mathcal{J}}^{I(p,q)} \arrow[ur, dashed] \arrow[dr, dashed] & \nparallel & \boxed{\mathcal{FSS} \cap \mathcal{J}^{\ell_2}} \arrow[ur, dashed] \arrow[dr] & \nparallel & \boxed{\mathcal{FSS} \vee \mathcal{J}^{\ell_2}} \arrow[r, dotted] & \boxed{L(\ell_p, \ell_q)} \\
 & & \boxed{\mathcal{J}^{I(2,q)}} \arrow[ur, dashed] & & \boxed{\mathcal{J}^{\ell_2}} \arrow[ur, dashed] & &
\end{tikzcd}$$

This solves Question (i) in [21] and shows that $\mathcal{J}^{I(p,q)}$ is different from $\mathcal{FSS} \cap \mathcal{J}^{\ell_2}$, and that the two (different) closed subideals $\mathcal{J}^{I(p,2)}$ and $\mathcal{J}^{I(2,q))}$ lie between them.

In order to show Theorem 3.1 we need to find two operators T and S in $\mathcal{FSS} \cap \mathcal{J}^{\ell_2}$, so that $T \in \mathcal{J}^{I(p,2)} \setminus \mathcal{J}^{I(2,q)}$ and $S \in \mathcal{J}^{I(2,q)} \setminus \mathcal{J}^{I(p,2)}$. We will first need the following result.

Theorem 3.2. *For every* $1 < r < \infty$ *there exists a constant* $K = K(r) > 0$ *and for all* $n \in \mathbb{N}$ *a number* $N = N(n, r) \in \mathbb{N}$*, such that every* N*–dimensional subspace* $F \subset \ell_r$ *contains an* n*–dimensional subspace* E *which is* K*–complemented in* ℓ_r *and* K*–isomorphic to* $\ell_2(n)$*.*

Remark 3.3. Theorem 3.2 follows from the finite dimensional version of Khintichin's Theorem (see [9, Theorem 6.28]). Better estimates on $N(n,r)$ and $K(r)$ can be obtained by applying simultaneously Dvoretzky's theorem both in a subspace $F \subset \ell_r$ and in its dual F^* (see e.g., [19]). This gives the result with $N = Cn^{r/2}$ and $K = C'\sqrt{\max\{r,r'\}}$, where $C, C' > 0$ are absolute constants. This theorem can be also viewed, for example, as a special case of results in [10].

Proof of Theorem 3.1. We will now construct the operators $T \in \mathcal{J}^{I(p,2)} \setminus \mathcal{J}^{I(2,q)}$ and $S \in \mathcal{J}^{I(2,q)} \setminus \mathcal{J}^{I(p,2)}$.

Put $C = \max(K(p), K(q))$ and for $n \in \mathbb{N}$ let $k_n = \max(N(p,n), N(q,n))$, where $K(p)$, $K(q)$, $N(p,n)$ and $N(q,n)$ are chosen as in Theorem 3.2. Using that result we can find for every $n \in \mathbb{N}$ a sequence $(x_{(n,i)})_{i=1}^n$ in $CB_{\ell_p(k_n)}$ so that

(4) $(x_{(n,i)})_{i=1}^n$ is C-equivalent to the unit vector basis of $\ell_2(n)$ and

(5) there is a projection P_n from $\ell_p(k_n)$ onto $\mathrm{span}(x_{(n,i)} : i = 1,2,\dots n)$ with $\|P_n\| \le C$.

For $n \in \mathbb{N}$ we define $I_n : \mathrm{span}(x_i^{(n)} : i = 1,2\dots,n) \to \ell_2(n)$, by $I_n(x_{(n,i)}) = e_{(2,n,i)}$, $i = 1,\dots n$. I_n is thus a C-isomorphism. Writing ℓ_p as ℓ_p-sum of $\ell_p(k_n)$and ℓ_2 as ℓ_2-sum of $\ell_2(n)$, $n \in \mathbb{N}$, we define $\tilde{S}$ as follows

$$\tilde{S} : \big(\oplus_{n=1}^\infty \ell_p(k_n)\big)_p \to \big(\oplus_{n=1}^\infty \ell_2(n)\big)_2, \quad (x_n) \mapsto \big(I_n \circ P_n(x_n) : n\in\mathbb{N}\big)$$

It follows that $\|\tilde{S}\| \le C^2$. Finally we let $S := I(2,q) \circ \tilde{S} \in \mathcal{J}^{I(2,q)}$.

The construction of $T : \ell_p \to \ell_q$ is similar. Using again Theorem 3.2 we find for each $n\in\mathbb{N}$ vectors $(y_{(n,i)} : i = 1,2\dots n)$ in $CB_{\ell_q(k_n)}$ so that

(6) $(y_{(n,i)})_{i=1}^n$ is C-equivalent to the unit vector basis of $\ell_2(n)$, and

(7) there is a projection Q_n from $\ell_q(k_n)$ onto $\mathrm{span}(y_{(n,i)} : i = 1,2,\dots n)$ with $\|Q_n\| \le C$.

Let $J_n : \ell_2(n) \to \ell_q(k_n)$, be the linear map which assigns to $e_{(2,n,i)}$ the vector $y_{(n,i)}$, $i = 1,2\dots n$, then J_n is a C-isomorphism onto its image, and by writing again ℓ_2 as ℓ_2-sum of $\ell_2(n)$ and ℓ_q as ℓ_q-sum of $\ell_q(k_n)$, $n \in \mathbb{N}$, we define $\tilde{T}$ as

$$\tilde{T} : \big(\oplus_{n=1}^\infty \ell_2(n)\big)_2 \to \big(\oplus_{n=1}^\infty \ell_q(k_n)\big)_q, \quad (x_n) \mapsto \big(J_n(x_n) : n\in\mathbb{N}\big).$$

Thus $\tilde{T}$ is a bounded operator with $\|\tilde{T}\| \le C$ and $T := \tilde{T} \circ I(p,2) \in \mathcal{J}^{I(p,2)}$.

In order to show that $S \notin \mathcal{J}^{I(p,2)}$ and $T \notin \mathcal{J}^{I(2,q)}$ we will find two functionals Φ and Ψ in $L^*(\ell_p,\ell_q)$ so that $\Phi(S) = 1$ and $\Phi|_{\mathcal{J}^{I(p,2)}} \equiv 0$, and, conversely $\Psi(T) = 1$ and $\Psi|_{\mathcal{J}^{I(2,q)}} \equiv 0$.

Let q' be the conjugate of q (i.e. $\frac{1}{q} + \frac{1}{q'} = 1$). For $n \in \mathbb{N}$ we define

$$\tilde{\Phi}_n : L(\ell_p(k_n), \ell_q(n)) \to \mathbb{R}, \qquad \text{with } \tilde{\Phi}_n(V) = \frac{1}{n}\sum_{i=1}^n \langle e_{(q',n,i)}, V(x_{(n_i)})\rangle.$$

Since by choice $\|x_{(n,i)}\| \le C$, for $i = 1,\dots,n$, it follows that $\|\tilde{\Phi}_n\| \le C$. We extend $\tilde{\Phi}_n$ in the canonical way to a functional in $L^*(\ell_p,\ell_q)$, i.e let $E_n : \ell_p(k_n) \to \ell_p = (\oplus_{n=1}^\infty \ell_p(k_n))$

be the canonical embedding to the n-component and let $F_n : \ell_q = (\oplus_{j=1}^{\infty} \ell_q(j)) \to \ell_q(n)$ be the projection onto the n-th component, for $n \in \mathbb{N}$ and put $\Phi_n(U) = \tilde{\Phi}_n(F_n \circ U \circ E_n)$ for $U \in L(\ell_p, \ell_q)$. Then also $\|\Phi_n\| \le C$ and we let $\Phi \in L^*(\ell_p, \ell_q)$ be a w* accumulation point of the sequence (Φ_n) in $L^*(\ell_p, \ell_q)$. Since $F_n \circ S \circ E_n(x_{(n,i)})$ is the i-th unit vector in $\ell_q(n)$ it follows that $\Phi(S) = \lim_{n\to\infty} \Phi_n(S) = 1$.

The definition of $\Psi \in L^*(\ell_p, \ell_q)$ is as follows. Since $(y_{(n,i)} : i = 1, 2, \ldots, n)$ is C-isomorphic to $(e_{(2,n,i)} : i = 1, 2 \ldots, n)$ and its linear span is C-complemented in $\ell_q(k_n)$, we can find a sequence $(y^*_{(n,i)} : i = 1, 2 \ldots, n) \subset \ell_{q'}(k_n)$, which is C- -isomorphic to $(e_{(2,n,i)} : i = 1, 2 \ldots, n)$, and satisfies $\langle y^*_{(n,i)}, y_{(n,j)} \rangle = \delta_{(i,j)}$ for $1 \le i, j \le n$.

For $n \in \mathbb{N}$ we can then write the projection $Q_n : \ell_q(k_n) \to \mathrm{span}(y_{(n,i)} : i = 1, 2, \ldots, n)$ (which was introduced in (7)) as

$$Q_n = \sum_{i=1}^{n} y_{(n,i)} \otimes y^*_{(n,i)} : \ell_q(k_n) \to \mathrm{span}(y_{(n,i)} : i = 1, 2, \ldots, n), \quad z \mapsto \sum_{i=1}^{n} y_{(n,i)} \langle y^*_{(n,i)}, z \rangle,$$

Then we define for $n \in \mathbb{N}$

$$\tilde{\Psi}_n : L(\ell_p(n), \ell_q(k_n)) \to \mathbb{R} \qquad \text{with } \tilde{\Psi}(U) = \frac{1}{n} \sum_{i=1}^{n} \langle y^*_{(n,i)}, U(e_{(p,n.i)}) \rangle.$$

We let Ψ_n be the canonical extension to a functional in $L^*(\ell_p, \ell_q)$, i.e. for $U \in L(\ell_p, \ell_q)$ we let $\Psi_n(U) = \tilde{\Psi}(F'_n \circ U \circ E'_n)$, where $E'_n : \ell_p(n) \to \ell_p = \big(\oplus_{j\in\mathbb{N}} \ell_p(j)\big)_p$, is the canonical embedding into the n-th component, and $F'_n : \big(\oplus_{j\in\mathbb{N}} \ell_q(k_j)\big)_q \to \ell_q(k_n)$ is the projection onto the n-th component. Since $\|y^*_{(n,i)}\|_{q'} \le C$, for $i = 1, 2 \ldots n$, it follows that $\|\Psi_n\| \le C$ and we let $\Psi \in L^*(\ell_p, \ell_q)$ be a w*-accumulation point of (Ψ_n). Since $T(e_{(p,n,i)}) = y_{(n,i)}$ for $i = 1, 2 \ldots, n$, it follows that $\Psi(T) = \lim_{n\to\infty} \langle \Psi_n, T \rangle = 1$.

It is left to show that $\mathcal{J}^{I(p,2)} \subset \ker(\Phi)$ and that $\mathcal{J}^{I(2,q)} \subset \ker(\Psi)$. To do so, we need a result which is of independent interest and will therefore be stated separately and more generally than needed. ☐

Definition 3.4. Let X be a finite or infinite dimensional Banach space with a normalized basis (e_i). We put for $j \in \mathbb{N}$, $j \le \dim(X)$, if X is infinite dimensional,

$$n_X(j) = \min\Big\{ \Big\| \sum_{i\in I} e_i \Big\| : I \subset \mathbb{N}, \#I = j \Big\}, \text{ and } N_X(j) = \max\Big\{ \Big\| \sum_{i\in I} e_i \Big\| : I \subset \mathbb{N}, \#I = j \Big\},$$

or, if $\dim(X) < \infty$,

$$n_X(j) = \min\Big\{ \Big\| \sum_{i\in I} e_i \Big\| : I \subset \{1, 2, \ldots \dim(X)\}, \#I = j \Big\}$$

and

$$N_X(j) = \max\Big\{\Big\|\sum_{i\in I} e_i\Big\| : I \subset \{1,2,\dots \dim(X)\}, \#I = j\Big\}.$$

Lemma 3.5. *Assume that E and F are two finite dimensional spaces, both having C_u-unconditional and normalized bases $(e_i : i = 1, 2 \dots m)$ and $(f_j : j = 1, \dots n)$, respectively.*

Assume further that there are $1 < t < s < \infty$ and positive constants c_1, and c_2, so that for all $\ell \in \mathbb{N}$

$$N_E(\ell) \le c_1 \ell^{1/s} \text{ and } n_F(\ell) \ge c_2 \ell^{1/t}. \tag{8}$$

Then there exists a number $c > 0$, depending only on s, t, c_u, c_1, and c_2, so that for every linear operator $T : E \to F$ and any $\rho > 0$.

$$|\{i \le m : ||T(x)||_\infty = \max_{j\le n} |f_j^*(T(e_i)) \ge \|T\|\rho\}| \le c\rho^{\frac{-s^2}{(s-1)(s-t)}}, \tag{9}$$

where (f_j^*) are the coordinate functionals to (f_j). Moreover, if $c_u = c_1 = c_2 = 1$, then we can choose $c = 1$.

Corollary 3.6. *Under the assumptions of Lemma 3.5, it follows that*

$$\frac{1}{m}\sum_{i=1}^m \|T(e_i)\|_\infty \le \|T\|(1+c)m^{-r(s,t)}, \text{ where } r(s,t) = \frac{(s-1)(s-t)}{(s-1)(s-t)+s^2}, \text{ for } s > t \ge 1. \tag{10}$$

Proof. First note that for any $\rho > 0$ Lemma 3.5 yields

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^m \|T(e_i)\|_\infty &= \frac{1}{m}\sum_{i=1, \|T(e_i)\|_\infty \le \rho\|T\|}^m \|T(e_i)\|_\infty + \frac{1}{m}\sum_{i=1, \|T(e_i)\|_\infty > \rho\|T\|}^m \|T(e_i)\|_\infty \\ &\le \|T\|\rho + c\|T\|\frac{\rho^{\frac{-s^2}{(s-1)(s-t)}}}{m}.\end{aligned}$$

Then we let

$$\rho = m^{-\frac{(s-1)(s-t)}{(s-1)(s-t)+s^2}},$$

which implies that

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^m \|T(e_i)\|_\infty &\le \|T\|m^{-\frac{(s-1)(s-t)}{(s-1)(s-t)+s^2}} + c\|T\|m^{-1}m^{\frac{(s-1)(s-t)}{(s-1)(s-t)+s^2}\frac{s^2}{(s-1)(s-t)}} \\ &= \|T\|m^{-\frac{(s-1)(s-t)}{(s-1)(s-t)+s^2}} + c\|T\|m^{-\frac{(s-1)(s-t)}{(s-1)(s-t)+s^2}} = (1+c)\|T\|m^{-r(s,t)}.\end{aligned}$$

□

Proof of Lemma 3.5. For the sake of a better readability we will assume that $c_1 = c_2 = c_u = 1$. The general case follows in the same way. We can also assume that $\|T\| = 1$.

Let $T : E \to F$ and write $y_i = T(e_i)$ as $y_i = \sum_{j=1}^n \beta(i,j) f_j$. Let $\rho > 0$ and put

$$A = A_\rho = \big\{ i \in \{1,2,\dots,m\} : \max |\beta(i,j)| \ge \rho \big\}.$$

For $i \in A$ choose $j_i \in \{1,2\dots,n\}$ so that $|\beta(i,j_i)| \ge \rho$. Let $\tilde{A} = \{j_i : i \in A\}$ and for $j \in \tilde{A}$ let $A_j = \{i \in A : j_i = j\}$. In order to estimate $|A_j|$ and then $\tilde{A}$ we compute

$$\begin{aligned}
|A_j|^{1/s} &\ge N_E(|A_j|) \quad \text{(By (8))}\\
&\ge \Big\| \sum_{i\in A_j} \operatorname{sign}(\beta(i,j)) e_j \Big\|_E\\
&\ge \Big\| T\Big(\sum_{i\in A_j} \operatorname{sign}(\beta(i,j)) e_j \Big) \Big\|_F \quad \text{(Since } \|T\| = 1)\\
&\ge \Big\langle f_j^*, \sum_{i\in A_j} T\Big(\sum_{i\in A_j} \operatorname{sign}(\beta(i,j)) e_j \Big) \Big\rangle\\
&= \sum_{i\in A_j} |\beta(i,j)| \ge |A_j| \rho
\end{aligned}$$

which yields $|A_j|^{1-\frac{1}{s}} \le \rho^{-1}$, and thus

$$|A_j| \le \rho^{-1/(1-\frac{1}{s})} = \rho^{-\frac{s}{s-1}}.$$

Since $|A| = \sum_{j\in\tilde{A}} |A_j| \le |\tilde{A}| \cdot \rho^{-\frac{s}{s-1}}$, we obtain

$$|\tilde{A}| \ge |A| \rho^{\frac{s}{s-1}}. \tag{11}$$

Let $(r_j)_{j=1}^m$ be a Rademacher sequence on some probability space $(\Omega, \Sigma, \mathbb{P})$, this means that $r_1, r_2, \dots r_m$ are independent and $\{\pm 1\}$-valued, with $\mathbb{P}(\{r_j = 1\}) = \mathbb{P}(\{r_j = -1\}) = 1/2$ for $j = 1, 2 \dots n$. We compute

$$\begin{aligned}
|A|^{1/s} &\ge N_E(|A|) \quad \text{(By (8))}\\
&\ge \mathbb{E}\Big(\Big\| \sum_{i\in A} r_i e_i \Big\|_E \Big)\\
&\ge \mathbb{E}\Big(\Big\| \sum_{i\in A} \sum_{j=1}^n r_i \beta(i,j) f_j \Big\|_F \Big) \quad \text{(Since } \|T\| \le 1)\\
&= \mathbb{E}\Big(\Big\| \sum_{j=1}^n f_j \Big| \sum_{i\in A} r_i \beta(i,j) \Big| \Big\|_F \Big) \quad \text{(By 1-unconditionality of } (f_j)).
\end{aligned}$$

Applying the multidimensional version of Jensen's inequality (c.f [7, 10.2.6, page 348]) to the convex function $\mathbb{R}^n \ni z \to \| \sum_{j=1}^n z_j f_j \|_F$ and the $\mathbb{R}^n$ valued random vector $Z = \big(| \sum_{i \in A} r_i \beta(i,j) | : j \le n \big)$ we obtain

$$\begin{aligned} |A|^{1/s} &\ge \Big\| \sum_{j=1}^n f_j \mathbb{E}\Big(\Big| \sum_{i\in A} r_i \beta(i,j) \Big| \Big) \Big\|_F \\ &\ge \Big\| \sum_{j\in \tilde{A}} f_j \mathbb{E}\Big(\Big| \sum_{i\in A} r_i \beta(i,j) \Big| \Big) \Big\|_F \quad \text{(By 1-uncondtionality of } (f_j)) \end{aligned}$$

For each $j \in \tilde{A}$ there is an $i_j \in A$ so that $|\beta(i,j)| \ge \rho$. Let r be anther ± 1 random variable with $\mathbb{P}(r = 1) = \mathbb{P}(r = -1) = 1/2$, which is independent to $(r_j : j = 1, \dots m)$ then

$$\begin{aligned} \mathbb{E}\Big(\Big| \sum_{i\in A} r_i \beta(i,j) \Big| \Big) &= \mathbb{E}\Big(\Big| r_{i_j} \beta(i_j,j) + r \sum_{i \in A \setminus \{i_j\}} r_i \beta(i,j) \Big| \Big) \\ &= \mathbb{E}\left(\frac{1}{2} \Big| r_{i_j} \beta(i_j,j) + \sum_{i \in A \setminus \{i_j\}} r_i \beta(i,j) \Big| + \frac{1}{2} \Big| r_{i_j} \beta(i_j,j) - \sum_{i \in A \setminus \{i_j\}} r_i \beta(i,j) \Big| \right) \\ &\ge \mathbb{E}(|r_{i_j} \beta(i_j,j)|) \quad \text{(Since } |a+b| + |a-b| \ge 2|a|) \\ &\ge \rho. \end{aligned}$$

Using again the 1-unconditionality of $(f_j : j = 1, 2 \dots n)$ we deduce therefore that

$$|A^{1/s}| \ge \Big\| \sum_{j\in \tilde{A}} f_j \mathbb{E}\Big(\Big| \sum_{i\in A} r_i \beta(i,j) \Big| \Big) \Big\|_F \ge \rho \Big\| \sum_{j \in \tilde{A}} f_j \Big\|_F \ge n_F(|\tilde{A}|),$$

and thus by our assumption (8) and by (11) we obtain

$$|A|^{1/s} \ge n_F(|\tilde{A}|) \ge |\tilde{A}|^{1/t} \ge |A|^{1/t} \rho^{\frac{s}{ts-t}}.$$

Solving for $|A|$ yields

$$|A| \le \rho^{-\frac{s}{ts-t}\frac{st}{s-t}} = \rho^{\frac{-s^2}{(s-1)(s-t)}},$$

which proves our claim. $\square$

Continuation of Proof of Theorem 3.1. In order to show that $\mathcal{J}^{I(p,2)} \subset \ker(\Phi)$ we let $A \in L_2(\ell_2, \ell_q)$ and $B \in L(\ell_p)$. We need to show that $\Phi(A \circ I(p,2) \circ B) = 0$. W.lo.g. we assume that $\|A\|, \|B\| \le 1$.

Consider $B'_n : \ell_2(n) \to \ell_p(k_n)$ with $B'(e_{(2,n,i)}) = B(x_{(n,i)})$, where we consider $\ell_p(k_n)$ canonically embedded into $\ell_p = \big(\oplus_{j=1}^\infty \ell(k_j) \big)$. Then $\|B'_n\| \le C$ and applying therefore Corollary 3.6 to B', $s = 2$ and $t = p$, we obtain

$$\frac{1}{n} \sum_{i=1}^n \|B(x_{(n,i)})\|_\infty = \frac{1}{n} \sum_{i=1}^n \|B'_n(e_{(2,n,i)})\|_\infty \le 2Cn^{-r(2,p)}.$$

which by the concavity of $[0,\infty)\ni\xi\mapsto\xi^{(2-p)/2}$ implies that

$$\frac{1}{n}\sum_{i=1}^{n}\|B(x_{(n,i)})\|_\infty^{(2-p)/2}\le\Big(\frac{1}{n}\sum_{i=1}^{n}\|B(x_{(n,i)})\|_\infty\Big)^{(2-p)/2}\le(2C)^{(2-p)/2}n^{-r(2,p)(2-p)/2}. \tag{12}$$

Secondly we observe that for any $i=1,2\dots n$

$$\begin{aligned}\|I(p,2)(B(x_{(n,i)}))\|_2&=\Big(\sum_{j=1}^{k_n}|B(x_{(n,i)})(j)|^2\Big)^{1/2}\\&=\Big(\sum_{j=1}^{k_n}|B(x_{(n,i)})(j)|^p|B(x_{(n,i)})(j)|^{2-p}\Big)^{1/2}\\&\le\|B(x_{(n,i)})\|_\infty^{(2-p)/2}\cdot\|B(x_{(n,i)})\|_p^{p/2}\le C^{p/2}\|B(x_{(n,i)})\|_\infty^{(2-p)/2}.\end{aligned} \tag{13}$$

It follows therefore that

$$\begin{aligned}\Big|\Phi_n(A\circ I(p,2)\circ B)\Big|&=\frac{1}{n}\Big|\sum_{i=1}^{n}\langle e_{(q',n,i)},A\circ I_{(p,2)}\circ B(x_{(n,i)})\rangle\Big|\\&=\frac{1}{n}\Big|\sum_{i=1}^{n}\langle A^*(e_{(q',n,i)}),I_{(p,2)}\circ B(x_{(n,i)})\rangle\Big|\\&\le\frac{1}{n}\sum_{i=1}^{n}\|A^*(e_{(q',n,i)})\|_2\|I_{(p,2)}\circ B(x_{(n,i)}\|_2\\&\le\|A^*\|C^{p/2}\frac{1}{n}\sum_{i=1}^{n}\|x_{(n,i)}\|_\infty^{(2-p)/2}\qquad\text{(By (13))}\\&\le C^{p/2}(2C)^{(2-p)/2}n^{-r(2,p)(2-p)/2}\to_{n\to\infty}0\qquad\text{(By (12))}.\end{aligned}$$

This implies that $\mathcal{J}^{I(p,2)}\subset\ker(\Phi)$.

In order to show that $\mathcal{J}^{I(2,q)}\subset\ker(\Psi)$, let $B\in L(\ell_p,\ell_2)$ and $A\in L(\ell_q)$ with $\|B\|,\|A\|\le1$. We need to show that $\Psi(A\circ I_{(2,q)}\circ B)=0$.

Let $A'_n:\ell_2(n)\to\ell_{q'}(k_n)$, defined by $A'_n(e^{(2,n,i)})=A^*(y^*_{(n,i)})$, $i=1,2\dots n$ (we consider $\ell_{q'}(k_n)$ in the canonical way as subspace of $\ell_{q'}=(\oplus_{j=1}^\infty\ell_{q'}(k_n))_q)$. It follows from the choice of $(y^*_{(n,i)}:i=1,2\dots n)$ that $\|A'_n\|\le C$ and from Corollary 3.6 (with $s=2$ and $t=q'$) we deduce that

$$\frac{1}{n}\sum_{i=1}^{n}\|A^*(y^*_{(n,i)})\|_\infty=\frac{1}{n}\sum_{i=1}^{n}\|A'(e_{(2,n,i)})\|_\infty\le2Cn^{-r(2,q')}.$$

Using the concavity of the function $[0,\infty)\ni\xi\to\xi^{(2-q')/2}$ we deduce

$$\frac{1}{n}\sum_{i=1}^{n}\|A^*(y^*_{(n,i)})\|_\infty^{(2-q')/2}=\Big(\frac{1}{n}\sum_{i=1}^{n}\|A^*(y^*_{(n,i)})\|_\infty\Big)^{(2-q')/2}\le(2C)^{(2-q')/2}n^{-r(2,q')(2-q')/2}. \tag{14}$$

It is easy to see that $I_{(q',2)}$ is the adjoint of $I_{(2,q)}$ and we compute for $i = 1, 2 \dots n$

$$\|I(q',2) \circ A^*(y^*(n,i))\|_2 = \Big(\sum_{j=1}^{k_n} \big(A^*(y^*(n,i))(j)\big)^2\Big)^{1/2} \tag{15}$$
$$= \Big(\sum_{j=1}^{k_n} \big|A^*(y^*(n,i))(j)\big|^{q'} \big|A^*(y^*(n,i))(j)\big|^{2-q'}\Big)^{1/2}$$
$$\leq \|A^*(y^*(n,i))\|_\infty^{(2-q')/2} \|y_{(n,i)}\|_{q'}^{q'/2}$$
$$\leq C^{q'/2} \|A^*(y^*(n,i))\|_\infty^{(2-q')/2}.$$

Therefore it follows

$$|\langle \psi_n, U\rangle| = \frac{1}{n}\Big|\sum_{i=1}^{n} \langle A \circ I_{(2,q)} \circ B(e_{(p,n,i)}), y^*(n,i)\rangle\Big|$$
$$= \frac{1}{n}\Big|\sum_{i=1}^{n} \langle B(e_{(p,n,i)}), I_{(q',2)} \circ A^*(y^*(n,i))\rangle\Big|$$
$$\leq \frac{1}{n}\sum_{i=1}^{n} \|B(e_{(p,n,i)})\|_2 \cdot \|I_{(q',2)} \circ A^*(y^*_{(n,i)})\|_2$$
$$\leq \|B\| C^{q'/2} \frac{1}{n} \sum_{i=1}^{n} \|A^*(y^*(n,i))\|_\infty^{(2-q')/2} \quad \text{(By (15))}$$
$$\leq C^{q'/2}(C+1)^{(2-q')/2} n^{-r(2,q')(2-q')/2} \to_{n\to\infty} 0 \quad \text{(By (14))}.$$

which implies our claim, and finishes the proof or Theorem 3.1. □

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