EMBEDDING UNIFORMLY CONVEX SPACES INTO SPACES WITH VERY FEW OPERATORS

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Abstract. We prove that every separable uniformly convex Banach space X embeds into a Banach space Z which has the property that all bounded linear operators on Z are compact perturbations of scalar multiples of the identity. More generally, the result holds for all separable reflexive Banach spaces of Szlenk index ω_0.

1. Introduction

The main goal of this paper is to show the following two results.

Theorem A. *Let X be a separable uniformly convex Banach space. Then X embeds in a Banach space Z, whose dual space is isomorphic to ℓ_1, and which has the property that all operators T on Z are of the form $T = \lambda Id + K$, where Id denotes the identity, λ is a scalar and K is a compact operator on Z.*

After introducing some terminology we will formulate a generalized version of Theorem A in Section 3.

The third named author and S. Argyros [AH] constructed a Banach space Z with the property that every operator on Z is a compact perturbation of a scalar multiple of the identity. Moreover Z^* is isomorphic to ℓ_1. In [FOS] it is proved that if X has a separable dual, then X embeds into a Banach space Y with Y^* isomorphic to ℓ_1. In proving Theorem A we will integrate arguments from both papers. Theorem A will in fact apply to all separable reflexive spaces of Szlenk index w_0.

It is perhaps worth pointing out that, just as in [AH], the space Z, constructed in the proof of Theorem A, will have some additional interesting properties. Namely,

i) Z is somewhat reflexive, i.e., every infinite dimensional subspace of Z contains an infinite dimensional reflexive subspace.
ii) $\mathcal{L}(Z)$ is amenable as a Banach algebra.
iii) $\mathcal{L}(Z)$ is separable.
iv) Every $T \in \mathcal{L}(Z)$ admits a non-trivial invariant subspace.

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The proof of Theorem A relies heavily on the Bourgain-Delbaen construction [BD]. The framework of which and the notation is reviewed in section 2. We present the construction in a manner so as to encompass previous B-D constructions. In Theorem 2.12 we give criteria that will ultimately yield the space Z, constructed in section 4 where Theorem A is proved, satisfies the "scalar plus compact" property. Section 3 contains more necessary preliminaries, reviewing embedding results and, in particular, the construction of [FOS].

Theorem B is proved in section 5. It was not known if a Banach space X, not containing c_0 and with separable dual, could be embedded into a space Y with a shrinking basis, also not containing c_0. Curiously, this is done also with the B-D construction and so we obtain in addition that Y^* is isomorphic to ℓ_1. The power of the B-D machinery is proving to be extraordinary.

Finally we note that Theorem A should extend to all separable reflexive Banach spaces using higher order Tsirelson spaces, embeddings in [OSZ2] and an appropriate modification of the arguments herein. We also note that in [AH] it is asked if Theorem A remains true for all spaces with separable dual.

2. The generalized Bourgain-Delbaen construction

In this section we lay out the general framework of the construction of *Bourgain-Delbaen spaces*. This framework is general enough to include the original space of Bourgain and Delbaen [BD], the spaces constructed in [AH], as well as the spaces constructed in [FOS]. We follow, with slight changes and some notational differences, the presentation in [AH] and start by introducing *Bourgain-Delbaen sets*.

Definition 2.1. (Bourgain-Delbaen-sets)

A sequence of finite sets $(\Delta_n : n\in\mathbb{N})$ is called a *Sequence of Bourgain - Delbaen Sets* if it satisfies the following recursive conditions:

Δ_1 is any finite set, and assuming that for some $n\in\mathbb{N}$ the sets Δ_1, $\Delta_2,\ldots$, Δ_n have been defined, we let $\Gamma_n=\bigcup_{j=1}^n \Delta_j$. We denote the unit vector basis of $\ell_1(\Gamma_n)$ by $(e^*_\gamma : \gamma\in\Gamma_n)$, and consider the spaces $\ell_1(\Gamma_j)$ and $\ell_1(\Gamma_n\setminus\Gamma_j)$, $j<n$, to be, in the natural way, embedded into $\ell_1(\Gamma_n)$.

For $n\ge 1$, Δ_{n+1} will then be the union of two sets $\Delta^{(0)}_{n+1}$ and $\Delta^{(1)}_{n+1}$, where $\Delta^{(0)}_{n+1}$ and $\Delta^{(1)}_{n+1}$ satisfy the following conditions.

The set $\Delta^{(0)}_{n+1}$ is finite and

$$\Delta^{(0)}_{n+1}\subset\big\{(n+1,\beta,b^*,f) : \beta\in(0,1], b^*\in B_{\ell_1(\Gamma_n)},\ \text{and}\ f\in V_{(n+1,\beta,b^*)}\big\}, \tag{1}$$

where $V_{(n+1,\beta,b^*)}$ is a finite set for $\beta\in(0,1]$ and $b^*\in B_{\ell_1(\Gamma_n)}$.

$\Delta^{(1)}_{n+1}$ is finite and

$$\Delta^{(1)}_{n+1}\subset\left\{(n+1,\alpha,k,\xi,\beta,b^*,f) : \begin{matrix}\alpha,\beta\in(0,1], k\in\{1,2,\ldots n-1\},\xi\in\Delta_k,\\ b^*\in B_{\ell_1(\Gamma_n\setminus\Gamma_k)}\ \text{and}\ f\in V_{(n+1,\alpha,k,\xi,\beta,b^*)}\end{matrix}\right\}, \tag{2}$$

where $V_{(n+1,\alpha,k,\xi,\beta,b^*)}$ is a finite set for $\alpha\in(0,1]$, $k\in\{1,2,\dots,n-1\}$, $\xi\in\Delta_k$, $\beta\in(0,1]$, and $b^*\in B_{\ell_1(\Gamma_n)}$.

If (Δ_n) is a sequence of Bourgain-Delbaen sets we put $\Gamma=\bigcup_{j=1}^\infty \Gamma_n$. For $n\in\mathbb{N}$, and $\gamma\in\Delta_n$ we call n the *rank of* γ and denote it by $\mathrm{rk}(\gamma)$. If $n\ge 2$ and $\gamma=(n,\beta,b^*,f)\in\Delta_n^{(0)}$, we say that γ is *of type* 0, and, in the case that $\gamma=(n,\alpha,k,\xi,\beta,b^*,f)\in\Delta_n^{(1)}$, we say that γ is *of type* 1. In both cases we call β *the weight of* γ and denote it by $\mathrm{wt}(\gamma)$ and f is called the *free variable,* which we denote by $\mathrm{f}(\gamma)$.

Given a sequence of Bourgain-Delbaen sets $\Delta=(\Delta_n:n\in\mathbb{N})$ we will always assume the sets $\Delta_n^{(0)}$, $\Delta_n^{(1)}$, Γ_n and Γ have been defined satisfying the conditions above. We consider the spaces $\ell_\infty(\bigcup_{j\in A}\Delta_j)$ and $\ell_1\big(\bigcup_{j\in A}\Delta_j\big)$, for $A\subset\mathbb{N}$, to be naturally embedded into $\ell_\infty(\Gamma)$ and $\ell_1(\Gamma)$, respectively.

We denote by $c_{00}(\Gamma)$ the real vector space of families $x=(x(\gamma):\gamma\in\Gamma)\subset\mathbb{R}$ for which the *support*, $\mathrm{supp}(x)=\{\gamma\in\Gamma:x(\gamma)\neq 0\}$, is finite. The unit vector basis of $c_{00}(\Gamma)$ is denoted by $(e_\gamma:\gamma\in\Gamma)$, or, if we think of c_{00} to be a subspace of a dual space, such as $\ell_1(\Gamma)$, by $(e^*_\gamma:\gamma\in\Gamma)$. If $\Gamma=\mathbb{N}$ we write c_{00} instead of $c_{00}(\mathbb{N})$.

Definition 2.2. (Bourgain-Delbaen families of functionals)

Assume that $(\Delta_n:n\in\mathbb{N})$ is a sequence of Bourgain-Delbaen sets. By induction on n we will define for all $\gamma\in\Delta_n$, elements $c^*_\gamma\in\ell_1(\Gamma_{n-1})$ and $d^*_\gamma\in\ell_1(\Gamma_n)$, with $d^*_\gamma=e^*_\gamma-c^*_\gamma$.

For $\gamma\in\Delta_1$ we define $c^*_\gamma=0$, and thus $d^*_\gamma=e^*_\gamma$.

Assume that for some $n\in\mathbb{N}$ we have defined $(c^*_\gamma:\gamma\in\Gamma_n)$, with $c^*_\gamma\in\ell_1(\Gamma_{j-1})$, if $j\le n$ and $\mathrm{rk}(\gamma)=j$. It follows therefore that $(d^*_\gamma:\gamma\in\Gamma_n)=(e^*_\gamma-c^*_\gamma:\gamma\in\Gamma_n)$ is a basis for $\ell_1(\Gamma_n)$ and thus for $k\le n$ we have the projections:

$$P^*_{(k,n]}:\ell_1(\Gamma_n)\to\ell_1(\Gamma_n),\quad \sum_{\gamma\in\Gamma_n}a_\gamma d^*_\gamma\to\sum_{\gamma\in\Gamma_n\setminus\Gamma_k}a_\gamma d^*_\gamma. \tag{3}$$

For $\gamma\in\Delta_{n+1}$ we then define

$$c^*_\gamma=\begin{cases}\beta b^* & \text{if } \gamma=(n+1,\beta,b^*,f)\in\Delta^{(0)}_{n+1},\\ \alpha e^*_\xi+\beta P^*_{(k,n]}(b^*) & \text{if } \gamma=(n+1,\alpha,k,\xi,\beta,b^*,f)\in\Delta^{(1)}_{n+1}.\end{cases} \tag{4}$$

We call $(c^*_\gamma:\gamma\in\Gamma)$, the *Bourgain-Delbaen family of functionals associated to* $(\Delta_n:n\in\mathbb{N})$. We will in this case consider the projections $P^*_{(k,n]}$ to be defined on all of $c_{00}(\Gamma)$, which is possible since $(d^*_\gamma:\gamma\in\Gamma)$ forms a vector basis of $c_{00}(\Gamma)$ and, (as we will observe later) under further assumptions, a Schauder basis of $\ell_1(\Gamma)$.

Remarks. The reason for using $*$ in the notation for $P^*_{(k,m]}$ is that later we will observe that the $P^*_{(k,m]}$ are the adjoints of some coordinate projections $P_{(k,m]}$ on a space Y with an FDD $\mathbf{F}=(F_j)$ onto $\oplus_{j\in(k,m]}F_j$.

Of course we could, in the definition of $\Delta^{(0)}_{n+1}$ and $\Delta^{(1)}_{n+1}$, assume $\beta=1$, rescale b^* accordingly, possibly increasing the number of free variables, then simply define $c^*_\gamma=b^*$, if γ is of type 0, or

$c^*=\alpha e^*_\xi+P^*_{(k,n]}b^*$, if γ is of type 1. Nevertheless, it will prove later more convenient to have this redundant representation which will allow us to change the weights of the elements of Γ and rescale the b^*'s, without changing the c^*_γ's. Moreover, it will be useful for recognizing that our framework is a generalization of the constructions in [AH], [BD] and [FOS].

The next observation is a combination of a result in [AH] and [FOS], the main idea tracing back to [BD].

Proposition 2.3. *Assume that $(\Delta_n : n\in\mathbb{N})$ is a sequence of Bourgain-Delbaen sets and let $(c^*_\gamma : \gamma\in\Gamma)$ be the corresponding family of associated functionals. Let $(P^*_{(k,m]} : k<m)$ and $(d^*_\gamma : \gamma\in\Gamma)$ be defined as in Definition 2.2. Thus*

$$P^*_{(k,n]} : c_{00}(\Gamma)\to c_{00}(\Gamma),\quad \sum_{\gamma\in\Gamma} a_\gamma d^*_\gamma \to \sum_{\gamma\in\Gamma_n\setminus\Gamma_k} a_\gamma d^*_\gamma.$$

For $n\in\mathbb{N}$ put $F^*_n=\operatorname{span}(d^*_\gamma : \gamma\in\Delta_n)$ and for $\theta\in[0,1/2)$ we define $C_1(\theta)=C_1=0$ and for $n\geq 2$,

$$C_n(\theta)=\sup\big\{\beta\|P^*_{(k,m]}(b^*)\| : \gamma=(\tilde n,\alpha,k,\xi,\beta,b^*,f)\in\Delta^{(1)}_{\tilde n}, k<m<\tilde n\leq n, \beta>\theta\big\},$$

with $\sup(\emptyset)=0$ and

$$C_n=C_n(0)=\sup\big\{\beta\|P^*_{(k,m]}(b^*)\| : \gamma=(\tilde n,\alpha,k,\xi,\beta,b^*,f)\in\Delta^{(1)}_{\tilde n}, k<m<\tilde n\leq n\big\}.$$

Then

$$\oplus_{j=1}^n F^*_j=\operatorname{span}(e^*_\gamma : \gamma\in\Gamma_n)=\ell_1(\Gamma_n), \tag{5}$$

and if $C=\sup_n C_n<\infty$, then $\mathbf{F}^*=(F^*_n)$ is an FDD for $\ell_1(\Gamma)$ whose decomposition constant M is not larger than $1+C$. Moreover, for $n\in\mathbb{N}$ and $\theta<1/2$,

$$C_n\leq\max\big(2\theta/(1-2\theta), C_n(\theta)\big). \tag{6}$$

Proof. As already noted, since $d^*_\gamma=e^*_\gamma-c^*_\gamma$, and $c^*_\gamma\in\ell_1(\Gamma_{n-1})$, for $n\in\mathbb{N}$ and $\gamma\in\Delta_n$, (5) holds. By induction on $n\in\mathbb{N}$ we will show that for all $0\leq m<n$, $\|P^*_{[1,m]}|_{\ell_1(\Gamma_n)}\|\leq 1+C_n$, and that (6) holds, whenever $\theta<1/2$. For $n=1$, and thus $m=0$ and $C_1=0$, the claim follows trivially ($\|P^*_\emptyset\|\equiv 0$). Assume the claim is true for some $n\in\mathbb{N}$. Using the induction hypothesis and the fact that every element of $B_{\ell_1(\Gamma_{n+1})}$ is a convex combination of $\{\pm e^*_\gamma : \gamma\in\Gamma_{n+1}\}$ and $C_n(\theta)\leq C_{n+1}(\theta)$, it is enough to show that for all $\gamma\in\Delta_{n+1}$ and all $m\leq n$

$$\|P^*_{[1,m]}(e^*_\gamma)\|\leq 1+C_{n+1} \text{ and} \tag{7}$$

$$\|\beta P^*_{(k,m]}(b^*)\|\leq\frac{2\theta}{1-2\theta}\vee C_n(\theta), \text{if } \beta\leq\theta<1/2 \text{ and } \gamma=(n+1,\alpha,k,\xi,\beta,b^*,f)\in\Delta^{(1)}_{n+1}. \tag{8}$$

According to (4) we can write

$$e^*_\gamma=d^*_\gamma+c^*_\gamma=d^*_\gamma+\alpha a^*+\beta P^*_{(k,n]}(b^*),$$

with $\alpha,\beta\in(0,1]$, $0\leq k<n$, $a^*\in B_{\ell_1(\Gamma_k)}$ (put $k=0$ and $a^*=0$ if γ is of type 0), and $b^*\in B_{\ell_1(\Gamma_n\setminus\Gamma_k)}$.

Thus

$$P^*_{[1,m]}(e^*_\gamma) = \alpha P^*_{[1,m]}(a^*) + \beta P^*_{(\min(m,k),m]}(b^*).$$

Now, if $k \ge m$, then (since $a^* \in \ell_1(\Gamma_k)$) $P^*_{[1,m]}(e^*_\gamma) = \alpha P^*_{[1,m]}(a^*)$ and thus our claim (7) follows from the induction hypothesis:

$$\|\alpha P^*_{[1,m]}(a^*)\| \le 1 + C_k \le 1 + C_{n+1}.$$

If $k < m$ it follows that

$$\|P^*_{[1,m]}(e^*_\gamma)\| \le \alpha\|a^*\| + \beta\|P^*_{(k,m]}(b^*)\| \le 1 + C_{n+1},$$

which implies (7).

In order to show (8), we deduce from the induction hypothesis for $\gamma = (n+1, \alpha, k, \xi, \beta, b^*, f) \in \Delta^{(1)}_{n+1}$, with $\beta \le \theta < 1/2$ that

$$\begin{aligned}
\|\beta P^*_{(k,m]}(b^*)\| &\le \beta(\|P^*_{[1,m]}|_{\ell_1(\Gamma_n)}\| + \|P^*_{[1,k]}|_{\ell_1(\Gamma_n)}\|)\\
&\le 2\theta(C_n + 1)\\
&\le \begin{cases} 2\theta\big(C_n(\theta)+1)\big) \le 2\theta C_n(\theta) + C_n(\theta)(1-2\theta) = C_n(\theta) & \text{if } C_n(\theta) > \frac{2\theta}{1-2\theta},\\ 2\theta\Big(\frac{2\theta}{1-2\theta}+1\Big) = \frac{2\theta}{1-2\theta} & \text{otherwise,}\end{cases}\\
&\le \max\Big(\frac{2\theta}{1-2\theta}, C_n(\theta)\Big).
\end{aligned}$$

which finishes the induction step and the proof of our claim. □

Remarks. Let Γ be linearly ordered as $(\gamma_j : j \in \mathbb{N})$ in such a way that $\text{rk}(\gamma_i) \le \text{rk}(\gamma_j)$, if $i \le j$. Then the same arguments show that, under the assumption $C < \infty$ stated in Proposition 2.3, $(d^*_{\gamma_j})$ is actually a Schauder basis of ℓ_1 [AH]. But for our purpose the FDD is the more natural coordinate system.

The spaces constructed in [AH] satisfy the condition that for some $\theta < 1/2$ we have $\beta \le \theta$, for all $\gamma = (n, \alpha, k, a^*, \beta, b^*, f) \in \Gamma$ of type 1. Thus in that case $C_n(\theta) = 0$, $n \in \mathbb{N}$, and the conclusion of Proposition 2.3 is true for $C \le 2\theta/(1-2\theta)$ and, thus $M \le 1/(1-2\theta)$. The spaces constructed in [FOS] satisfy the following condition: there is a $\theta < 1/2$ so that for all $\gamma = (n+1, \alpha, k, \xi, \beta, b^*, f) \in \Gamma$ of type 1, either $\beta \le \theta$ or $b^* = e^*_\eta$, for some $\eta \in \Gamma$ with $c^*_\eta = 0$ (and thus $d^*_\eta = e^*_\eta$). It follows therefore in the latter case that $P^*_{(k,m]}(b^*) = b^*$ or $P^*_{(k,m]}(b^*) = 0$, for $k \le m$, and, thus, that $C_n(\theta) \le 1$, for all n, and we conclude from Proposition 2.3 that $C \le \max(1, 2\theta/(1-2\theta))$ and thus $M \le \max(2, 1/(1-2\theta))$.

Assume we are given a sequence of Bourgain Delaben sets $(\Delta_n : n \in \mathbb{N})$, which satisfy the assumptions of Proposition 2.3 with $C < \infty$ and let M be the decomposition constant of the FDD (F^*_n) in $\ell_1(\Gamma)$. We now define the *Bourgain-Delbaen space associated to* $(\Delta_n : n \in \mathbb{N})$. For a finite or cofinite set $A \subset \mathbb{N}$ we let P^*_A be the projection onto the subspace $\oplus_{j\in A} F^*_j$ of $\ell_1(\Gamma)$ given by

$$P^*_A : \ell_1(\Gamma) \to \ell_1(\Gamma), \qquad \sum_{\gamma\in\Gamma} a_\gamma d^*_\gamma \mapsto \sum_{\gamma\in A} a_\gamma d^*_\gamma.$$

If $A=\{m\}$, for some $m\in\mathbb{N}$, we write P^*_m instead of $P^*_{\{m\}}$. For $m\in\mathbb{N}$ we denote by R_m the restriction operator from $\ell_1(\Gamma)$ onto $\ell_1(\Gamma_m)$ (in terms of the basis (e^*_γ)) as well the usual restriction operator from $\ell_\infty(\Gamma)$ onto $\ell_\infty(\Gamma_m)$. Since $R_m\circ P^*_{[1,m]}$ is a projection from $\ell_1(\Gamma)$ onto $\ell_1(\Gamma_m)$, for $m\in\mathbb{N}$, it follows that the map

$$J_m:\ell_\infty(\Gamma_m)\to\ell_\infty(\Gamma),\quad x\mapsto P^{**}_{[1,m]}\circ R^*_m(x),$$

is an isomorphic embedding ($P^{**}_{[1,m]}$ is the adjoint of $P^*_{[1,m]}$ and, thus, defined on $\ell_\infty(\Gamma)$). Since R^*_m is the natural embedding of $\ell_\infty(\Gamma_m)$ into $\ell_\infty(\Gamma)$ it follows for all $m\in\mathbb{N}$ that

(9) $\quad R_m\circ J_m(x)=x,$ for $x\in\ell_\infty(\Gamma_m)$, thus J_m is an extension operator,

(10) $\quad J_n\circ R_n\circ J_m(x)=J_m(x),$ whenever $m\le n$ and $x\in\ell_\infty(\Gamma_m)$,

and by Proposition 2.3,

(11) $\quad \|J_m\|\le M.$

Hence the spaces $Y_m=J_m(\ell_\infty(\Gamma_m))$, $m\in\mathbb{N}$, are finite-dimensional nested subspaces of $\ell_\infty(\Gamma)$ which (via J_m) are M-isomorphic images of $\ell_\infty(\Gamma_m)$. Therefore

(12) $$Y=\overline{\bigcup_{m\in\mathbb{N}}Y_n}^{\ell_\infty}$$

is a $\mathcal{L}_{M,\infty}$ space. We call Y the *Bourgain-Delbaen space associated to* (Δ_n).

Define for $m\in\mathbb{N}$

$$P_{[1,m]}:Y\to Y,\quad x\mapsto J_m\circ R_m(x).$$

We claim that $P_{[1,m]}$ coincides with the restriction of the adjoint $P^{**}_{[1,m]}$ of $P^*_{[1,m]}$ to the space Y. Indeed, if $n\in\mathbb{N}$, with $n\ge m$, and $x=J_n(\tilde{x})\in Y_n$, and $b^*\in\ell_1(\Gamma)$ we have that

$$\begin{aligned}\langle P^{**}_{[1,m]}(x),b^*\rangle&=\langle x,P^*_{[1,m]}(b^*)\rangle\\&=\langle R_m(x),R_m\circ P^*_{[1,m]}(b^*)\rangle \text{ (since } P^*_{[1,m]}(b^*)\in\operatorname{span}(e^*_\gamma:\gamma\in\Gamma_m))\\&=\langle P^{**}_{[1,m]}\circ R^*_m\circ R_m(x),b^*\rangle=\langle P_{[1,m]}(x),b^*\rangle.\end{aligned}$$

Thus our claim follows since $\bigcup_n Y_n$ is dense in Y.

We therefore deduce that Y has an FDD (F_m), with $F_m=(P_{[1,m]}-P_{[1,m-1]})(Y)$ and $Y_m=\oplus_{j=1}^m F_j\sim_M\ell_\infty(\Gamma_m)$ for $m\in\mathbb{N}$. Moreover, denoting by P_A the coordinate projections from Y onto $\oplus_{j\in A}F_j$, for all finite or cofinite sets $A\subset\mathbb{N}$, it follows that P_A is the adjoint of P^*_A restricted to Y, and P^*_A is the adjoint of P_A restricted to the subspace of Y^* generated by the F^*_n's.

Denote by $\|\cdot\|_*$ the dual norm of Y^* restricted to the sub space $\oplus_{j=1}^\infty F^*_j=\ell_1$. We claim that for all $b^*\in\ell_1(\Gamma)$

(13) $$\|b^*\|_*\le\|b^*\|_{\ell_1}\le M\|b^*\|_*.$$

The first inequality follows from the fact that $\|e^*_\gamma\|_* \le \|e^*_\gamma\|_{\ell^*_\infty} = 1$, for $\gamma\in\Gamma$, and the triangle inequality. To show the second inequality we let $b^*\in\ell_1(\Gamma_n)$ for some $n\in\mathbb{N}$ and choose $x\in S_{\ell_\infty(\Gamma_n)}$ so that $\langle b^*, x\rangle = \|b^*\|_{\ell_1}$. Then it follows from (11) and (9)

$$\|b^*\|_* \ge \Big\langle b^*, \frac{1}{M}J_n(x)\Big\rangle = \frac{1}{M}\|b\|_{\ell_1}.$$

We now recall some the notation introduced in [AH]. Assume that we are given a Bourgain-Delaben sequence (Δ_n) and the corresponding Bourgain-Delbaen family $(c^*_\gamma : \gamma\in\Gamma)$ and Bourgain-Delbaen space Y, which admits a decomposition constant $M<\infty$. As above we denote its FDD by (F_n). For $n\in\mathbb{N}$ and $\gamma\in\Delta_n$, we write

$$e^*_\gamma = d^*_\gamma + c^*_\gamma = d^*_\gamma + \begin{cases} \beta b^* & \text{if } \gamma=(n,\beta,b^*,f)\in\Delta^{(0)}_n,\\ e^*_\xi + \beta P^*_{(k,\infty)}(b^*) & \text{if } \gamma=(n,k,\xi,\beta,b^*,f)\in\Delta^{(1)}_n.\end{cases}$$

In the second case, we can write $e^*_\xi = d^*_\xi + c^*_\xi$, and, then we can insert for c^*_ξ its definition. We can proceed this way and eventually arrive (after finitely many steps) to a functional of type 0. By an easy induction argument we therefore deduce the following

Proposition 2.4. *For all $n\in\mathbb{N}$ and $\gamma\in\Delta_n$, there are $a\in\mathbb{N}$, $\beta_1,\beta_2,\dots\beta_a\in(0,1]$, numbers $0=p_0<p_1<p_2-1<p_2<p_3<p_3-1,\dots<p_{a-1}<p_a-1<p_a=n$ in $\mathbb{N}_0$, vectors b^*_j, $j=1,2\dots a$, with $b^*_j\in B_{\ell_1}(\Gamma)\cap \mathrm{span}(e^*_\eta : \eta\in\Gamma_{p_j-1}\setminus\Gamma_{p_{j-1}})$, and $(\xi_j)\subset\Gamma_n$, with $\xi_j\in\Delta_{p_j}$, for $j=1,2\dots a$, and $\xi_a=\gamma$, so that*

$$e^*_\gamma = \sum_{j=1}^a d^*_{\xi_j} + \beta_j P^*_{(p_{j-1},\infty)}(b^*_j) = \sum_{j=1}^a d^*_{\xi_j} + \beta_j P^*_{(p_{j-1},p_j)}(b^*_j). \tag{14}$$

We call the representation in (14) *the analysis of* γ and define for $\gamma\in\Gamma$, $\mathrm{age}(\gamma)=a$ to be the *age of* γ. We let the *sets cuts of* γ to be $\mathrm{cuts}(\gamma)=\{p_1,p_2,\dots p_a\}$. We put $\overline{\mathrm{w}}(\gamma)=\max_{j\le \mathrm{age}(\gamma)}\beta_j = \max_{j\le\mathrm{age}(\gamma)}\mathrm{wt}(\xi_j)$, and $\underline{\mathrm{w}}(\gamma)=\min_{j\le\mathrm{age}(\gamma)}\beta_j=\max_{j\le\mathrm{age}(\gamma)}\mathrm{wt}(\xi_j)$, and call these numbers the *maximal weight of* γ, and the *minimal weight of* γ, respectively.

Remark. Assume that $\gamma\in\Gamma$ and that $a=\mathrm{age}(\gamma)\in\mathbb{N}$, $\beta_1,\beta_2,\dots,\beta_a\in(0,1]$, $0=p_0<p_1<p_2-1<p_2<p_3<p_3-1,\dots<p_{a-1}<p_a-1<p_a=n$ in $\mathbb{N}_0$, vectors b^*_j, $j=1,2\dots a$, with $b^*_j\in B_{\ell_1}(\Gamma)\cap\mathrm{span}(e^*_\eta:\eta\in\Gamma_{p_j-1}\setminus\Gamma_{p_{j-1}})$, and $(\xi_j)\subset\Gamma_n$, with $\xi_j\in\Delta_{p_j}$, for $j=1,2\dots a$, and $\xi_a=\gamma$, are chosen so that (14) holds. Then for $r=1,2\dots a$ we can write

$$e^*_\gamma = e^*_{\gamma_r} + \sum_{j=r+1}^a d^*_{\xi_j} + \beta_j P^*_{(p_{j-1},\infty)}(b^*_j) = \sum_{j=1}^a d^*_{\xi_j} + \beta_j P^*_{(p_{j-1},p_j)}(b^*_j). \tag{15}$$

We call the representation (15) of e^*_γ a *partial analysis of* e^*_γ.

We need one more definition concerning the weights of the elements of Γ.

Definition 2.5. Let (Δ_n) be a sequence of Bourgain-Delbaen sets. We say that the *weights of all* $\gamma \in \Gamma$ *are comparable* if

$$q_0 = \inf_{\gamma\in\Gamma} \frac{\overline{\mathrm{w}}(\gamma)}{\underline{\mathrm{w}}(\gamma)} > 0.$$

Remark. The comparability of weights is only a property of the sets $\Delta_n^{(0)}$ and $\Delta_n^{(1)}$, $n\in\mathbb{N}$, but not of the functionals c_γ^*, $\gamma \in \Gamma$, and thus not a property of the corresponding Bourgain-Delbaen space Y. Indeed, Let $0 < c \leq 1$. By replacing $\gamma = (n, \beta, b^*, v) \in \Delta_n^{(0)}$ by $\gamma' = (n, c, \frac{\beta}{c} b^*, (v, \beta))$ and $\gamma = (n, \alpha, k, \xi, \beta, b^*, v) \in \Delta_n^{(0)}$ by $\gamma' = (n, \alpha, k, \xi, c, \frac{\beta}{c} b^*, (v, \beta))$, in the case that $\beta \leq c$, we convert the sets Δ_n into Bourgain-Delbaen sets of comparable weights, $q_0 \geq 1/c$, without changing the c_γ^*'s and, thus without changing the corresponding Bourgain-Delbaen space Y.

Definition 2.6. Let $x\in Y$. The *range of* x is the minimal interval $[p, q] \subset \mathbb{N}$, or $[p, \infty)$, so that $x \in \oplus_{j=p}^{q} F_j$, or $x \in \oplus_{j=p}^{\infty} F_j$, respectively. We denote the range of x by $\mathrm{rg}(x)$. We say that x has finite range if $\mathrm{rg}(x)$ is bounded.

The *local support of* x is defined to be the set

$$\mathrm{supp}_{\mathrm{loc}}(x) = \Big\{\gamma : \gamma \in \bigcup_{j\in\mathrm{rg}(x)} \Delta_j \text{ and } x(\gamma) \neq 0\Big\}$$

(recall that $x(\gamma)$, $\gamma \in \Gamma$, denotes the coordinates in $\ell_\infty(\Gamma)$, thus $x(\gamma) = e_\gamma^*(x)$).

A set $A \subset Y$ is called *of bounded local weight* if

$$\inf \Big\{\overline{\mathrm{w}}(\gamma) : \gamma \in \bigcup_{x\in A} \mathrm{supp}_{\mathrm{loc}}(x)\Big\} > 0.$$

A sequence $(x_n) \subset Y$ has *decreasing local weight* if

$$\lim_{n\to\infty} \sup\{\overline{\mathrm{w}}(\gamma) : \gamma \in \mathrm{supp}_{\mathrm{loc}}(x_n)\} = 0.$$

Remark. If (x_n) is a block sequence of decreasing local weight we can pass to a subsequence (x'_n) so that $U_1 \geq L_1 > U_2 \geq L_2 > U_3 \dots$ where

$$L_n = \inf_{\eta\in\mathrm{supp}_{\mathrm{loc}}(x'_n)} \underline{\mathrm{w}}(\eta) \text{ and } U_n = \sup_{\eta\in\mathrm{supp}_{\mathrm{loc}}(x'_n)} \overline{\mathrm{w}}(\eta) \text{ for } n\in\mathbb{N}.$$

If the weights of (Δ_n) are comparable and $A \subset Y$ has bounded local weights, then

$$\inf \Big\{\underline{\mathrm{w}}(\gamma) : \gamma \in \bigcup_{x\in A} \mathrm{supp}_{\mathrm{loc}}(x)\Big\} > 0.$$

Proposition 2.7. [AH] *Assume that the analysis of* $\gamma \in \Gamma$ *is given by*

$$e_\gamma^* = \sum_{j=1}^{a} d_{\xi_j}^* + \beta_j P_{(p_{j-1},\infty)}^*(b_j^*) = \sum_{j=1}^{a} d_{\xi_j}^* + \beta_j P_{(p_{j-1},p_j)}^*(b_j^*).$$

and assume that $x\in Y$ *is such that* $\{\xi_j : j = 1, 2, \dots a\} \cap \mathrm{supp}_{\mathrm{loc}}(x) = \emptyset$. *Then*

$$|x(\gamma)| \leq 2M\overline{\mathrm{w}}(\gamma)\|x\|.$$

Proof. If $\mathrm{rk}(\gamma) \le \max(\mathrm{rg}(x))$, then $x(\gamma) = 0$ (using that $e^*_\gamma = e^*_{\xi_a}$ and the assumption that $\{\xi_j : j = 1, 2, \dots a\} \cap \mathrm{supp}_{\mathrm{loc}}(x) = \emptyset$). If $\mathrm{rk}(\gamma) > \max(\mathrm{rg}(x))$, then there is an $s \in \{0, 1, 2 \dots, a-1\}$ so that $p_s \le \max \mathrm{rg}(x) < p_{s+1}$. It follows from (15) that ($p_0 = 0$ and $e^*_{\xi_0} := 0$)

$$\begin{aligned} |x(\gamma)| = |e^*_\gamma(x)| &= \Big| e^*_{\xi_s}(x) + \sum_{j=s+1}^{a} \beta_j P^*_{(p_{j-1},p_j)}(b^*_j)(x) + d^*_{\xi_j}(x) \Big| \\ &\le \Big| \overline{\mathrm{w}}(\gamma) P^*_{(p_s,p_{s+1})}(b^*_j)(x) \Big| \le 2M \overline{\mathrm{w}}(\gamma) \|x\|. \end{aligned}$$

□

Proposition 2.8. *Let $T : Y \to W$ be a bounded linear operator, W being a Banach space. Then $\|T(x_k)\| \to 0$ whenever (x_k) is a bounded block basis if and only if $\|T(x_k)\| \to 0$ whenever (x_k) is a bounded block basis of bounded local weight or a bounded block basis of decreasing local weight.*

Proof. Assume that the image of each block basis of bounded local weight is norm null and assume that (x_n) is a normalized block sequence in Y and $\varepsilon_0 > 0$, so that $\|T(x_n)\| \ge \varepsilon_0$, for all $n \in \mathbb{N}$. We show that there is bounded block sequence (z_n) with decreasing local weight so that $\inf_{n\in\mathbb{N}} \|T(z_n)\| > 0$.

Let $[p_n, q_n] = \mathrm{rg}(x_n)$, and put $u_n = x_n|_{\Gamma_{q_n}}$. Thus $\|u_n\|_{\ell_\infty(\Gamma_n)} \le 1$ and $x_n = J_{q_n}(u_n))$, for $n \in \mathbb{N}$. For $N \in \mathbb{N}$, $n \in \mathbb{N}$ and $\gamma \in \Gamma_{q_n}$ define

$$v_n^{(N)}(\gamma) = \begin{cases} u_n(\gamma) & \text{if } \overline{\mathrm{w}}(\gamma) \ge \frac{1}{N}, \\ 0 & \text{if } \overline{\mathrm{w}}(\gamma) < \frac{1}{N}, \end{cases} \text{ and } w_n^{(N)}(\gamma) = u_n(\gamma) - v_n^{(N)}(\gamma) = \begin{cases} u_n(\gamma) & \text{if } \overline{\mathrm{w}}(\gamma) < \frac{1}{N}, \\ 0 & \text{if } \overline{\mathrm{w}}(\gamma) \ge \frac{1}{N}. \end{cases}$$

and write $v_n^{(N)} = (v_n^{(N)}(\gamma) : \gamma \in \Gamma_{q_n})$, $y_n^{(N)} = J_{q_n}(v_n^{(N)}) \in Y$, $w_n^{(N)} = (w_n^{(N)}(\gamma) : \gamma \in \Gamma_{q_n})$, and $z_n^{(N)} = J_{q_n}(w_n^{(N)}) \in Y$. It follows that $\|y_n^{(N)}\|, \|z_n^{(N)}\| \le M$, by (11), and $x_n = y_n^{(N)} + z_n^{(N)}$ for all $n, N \in \mathbb{N}$.Thus, $(y_n^{(N)} : n \in \mathbb{N})$ and $(z_n^{(N)} : n \in \mathbb{N})$ are bounded block bases for all $N \in \mathbb{N}$. Moreover $(y_n^{(N)} : n \in \mathbb{N})$ is of bounded local weight, for all $N \in \mathbb{N}$, and using our assumption as well as an easy diagonalization argument we can assume, after passing to subsequences, if necessary, that $\|T(y_n^{(N)})\| \le \varepsilon_0/2$ whenever $N \le n$. But this implies that $\|T(z_n^{(n)})\| \ge \varepsilon_0/2$ and $(z_n^{(n)})$ is, by definition, of decreasing local weight. □

Definition 2.9. Let (x_n) be a block basis in Y, $\mathbf{w} = (w_i) \subset (0, 1]$ a decreasing null sequence and $C > 0$. We say (x_n) is a $(\mathbf{w}, C)$-*Rapidly Increasing Sequence*, or $(\mathbf{w}, C)$-RIS, if for $k \in \mathbb{N}$

(16) $\quad \|x_k\| \le C$ and $|x_k(\gamma)| \le C\overline{\mathrm{w}}(\gamma)$ if $k \ge 2$ and $\gamma \in \Gamma$ with $\underline{\mathrm{w}}(\gamma) \ge w_{\max \mathrm{rg}(x_{k-1})}$.

It is easy to see that Rapidly Increasing Sequences satisfy the following permanence properties.

Proposition 2.10. *Let $\mathbf{w} = (w_n : n \in \mathbb{N})$ be a decreasing null sequence in $(0, 1]$ and $C > 0$.*

a) *Every subsequence of a $(\mathbf{w}, C)$-RIS is a $(\mathbf{w}, C)$-RIS.*

b) *If* (x_n) *and* (y_n) *are* $(\mathbf{w},C)$*-RIS's and* $\alpha,\beta>0$*, then there is a subsequence* $(m_j)\subset\mathbb{N}$ *so that* $(\alpha x_{m_j}+\beta y_{m_j})_{j\in\mathbb{N}}$*, is a* $(\mathbf{w},(\alpha+\beta)C)$*-RIS.*

Proposition 2.11. *Assume that* (Δ_n) *has comparable weights. Let* (x_n) *be a bounded block sequence in* Y*, and assume that* (x_n) *is either of bounded local weight or has decreasing local weight. Let* $\mathbf{w}=(w_j)\subset(0,1]$*.*

Then there is a $C>0$ *so that* (x_n) *has a subsequence which is a* $(\mathbf{w},C)$*-RIS.*

Proof. First assume that (x_n) has decreasing local weight. By the remark after Definition 2.6 we can assume that $U_1\ge L_1>U_2\ge L_2\dots$

$$L_n=\inf_{\eta\in\mathrm{supp}_{\mathrm{loc}}(x_n)}\underline{\mathrm{w}}(\eta)\text{ and }U_n=\sup_{\eta\in\mathrm{supp}_{\mathrm{loc}}(x_n)}\overline{\mathrm{w}}(\eta)\text{ for }n\in\mathbb{N}.$$

By passing again to a subsequence we can assume that $w_{\max\mathrm{rg}(x_{k-1})}>U_k$, if $k\ge2$.

Assume that $k\in\mathbb{N}$, $k\ge2$, and $\gamma\in\Gamma$ with $\underline{\mathrm{w}}(\gamma)\ge w_{\max\mathrm{rg}x_{k-1}}>U_k$ and write e^*_γ, as in Proposition 2.4, by

$$e^*_\gamma=\sum_{j=1}^a d^*_{\xi_j}+\beta_j P^*_{(p_{j-1},p_j)}(b^*_j).$$

Since $[\underline{\mathrm{w}}(\xi_j),\overline{\mathrm{w}}(\xi_j)]\subset[\underline{\mathrm{w}}(\gamma),\overline{\mathrm{w}}(\gamma)]\subset(U_k,1]$, for all $1\le j\le a$), it follows that $\mathrm{supp}_{\mathrm{loc}}(x_k)\cap\{\xi_1,\xi_2,\dots\xi_a\}=\emptyset$. Thus Proposition 2.7 it follows that $|x_k(\gamma)|\le 2M\|x_k\|\overline{\mathrm{w}}(\gamma)$ for $\gamma\in\Gamma$. Our claim follows in this case by taking $C=2M\sup_{k\in\mathbb{N}}\|x_k\|$.

If (x_k) is of bounded local weight we let $w=\inf_{n\in\mathbb{N},\gamma\in\mathrm{supp}_{\mathrm{loc}}(x_n)}\underline{\mathrm{w}}(\gamma)$, which by the remark after Definition 2.9, is a positive number, and choose $C=\sup_{n\in\mathbb{N}}\|x_n\|2Mw^{-1}$. Then for all $\gamma\in\Gamma$ (note that in the second case below Proposition 2.7 applies)

$$|x_k(\gamma)|\le\begin{cases}\|x_k\| & \text{if }\underline{\mathrm{w}}(\gamma)\ge w\\ |x_k(\gamma)| & \text{if }\underline{\mathrm{w}}(\gamma)<w\end{cases}\le\left.\begin{cases}\overline{\mathrm{w}}(\gamma)w^{-1}\|x_k\| & \text{if }\underline{\mathrm{w}}(\gamma)\ge w\\ 2M\overline{\mathrm{w}}(\gamma)\|x_k\| & \text{if }\underline{\mathrm{w}}(\gamma)<w\end{cases}\right\}\le C\overline{\mathrm{w}}(\gamma)$$

which proves our claim in the second case. □

We finish this section by stating a criterion which implies that all operators $T:Y\to Y$ are compact perturbations of a multiplication operator. Most of the proof is based on the proof of a similar statement in [AH].

Theorem 2.12. *Let* (Δ_n) *be a sequence of Bourgain-Delbaen sets, with finite decomposition constant* M *and comparable weights. Assume furthermore that the FDD* (F_n) *of* Y*, which we defined above is shrinking, or equivalently that* Y^* *is isomorphic to* ℓ_1 *(condition sufficient for this will be given in the next section).*

Let X *be a reflexive subspace of* Y *with an FDD* (E_j)*, which has the property that* $E_j\subset\oplus_{i=m_{j-1}+1}^{m_j}F_i$*, for some increasing sequence* (m_j) *(i.e. every block sequence in* X *with respect to* (E_j) *is also a block sequence in* Y *with respect to* (F_j)*).*

Assume that $T : Y \to Y$ is a bounded linear operator satisfying the following condition:

(17) *There is a decreasing null sequence $\mathbf{w} = (w_j)$, so that for all $C < \infty$ and every $(\mathbf{w}, C)$-RIS (x_n) $\liminf_{n\to\infty} \operatorname{dist}(T(x_n), [x_n] + X) = 0$.*

Then there is a $\lambda \in \mathbb{R}$ and a compact operator K on Y so that $T = \lambda Id + K$.

Remark. As the proof will show the statement of Theorem 2.12 can be generalized as follows. Assume that there is set $\mathcal{S}$ of bounded block sequences (x_n) with the following properties:

a) All block sequences with uniformly bounded local weight and all sequences with decreasing local weight admit a subsequence which is in $\mathcal{S}$.
b) Every subsequence of a sequence in $\mathcal{S}$ is also in $\mathcal{S}$.
c) If $(x_n), (y_n)$ are in $\mathcal{S}$, then there is a subsequence (m_n) of $\mathbb{N}$ so that $(x_{m_n} + y_{m_n})$ is in $\mathcal{S}$.

Then, if $T : Y \to Y$ is a bounded linear operator satisfying the condition:

$$\liminf_{n\to\infty} \operatorname{dist}(T(x_n), [x_n] + X) = 0 \text{ whenever } (x_n) \in \mathcal{S},$$

there exists a $\lambda \in \mathbb{R}$ and a compact operator K so that $T = \lambda Id + K$.

Proof of Theorem 2.12. Assume that (x_n) is a $(\mathbf{w}, C)$ RIS which is seminormalized in the quotient space Y/X. By our assumption we can choose a subsequence (x'_n) of (x_n) and a bounded sequence $(\lambda_n) \subset \mathbb{R}$ so that $\lim_{n\to\infty} \|T(x'_n) - \lambda_n x'_n\|_{Y/X} = 0$, after passing again to a subsequence we can assume that $\lambda = \lim_{n\to\infty} \lambda_n$ exists and, thus, that $\lim_{n\to\infty} \|T(x'_n) - \lambda x'_n\|_{Y/X} = 0$.

Secondly, we claim that λ does not depend on (x_n), and that there is a universal $\lambda \in \mathbb{R}$ so that for all $C > 0$ every $(\mathbf{w}, C)$ RIS (x_n), which is seminormalized in Y/X, has a subsequence (x'_n) so that $\lim_{n\to\infty} \|T(x'_n) - \lambda x'_n\|_{Y/X} = 0$. Indeed, assume that (x_n) and (y_n) are such sequences, and assume that λ and μ are in $\mathbb{R}$ so that for some subsequences (x'_n) and (y'_n) of (x_n) and (y_n), respectively, it follows that $\lim_{n\to\infty} \|T(x'_n) - \lambda x'_n\|_{Y/X} = 0$ and $\lim_{n\to\infty} \|T(y'_n) - \mu y'_n\|_{Y/X} = 0$. Using Proposition 2.10 we can, after passing to subsequences, if necessary, assume that $\max \operatorname{rg}(x'_n) < \min \operatorname{rg}(y'_n) < \min \operatorname{rg}(x'_{n+1})$, for $n \in \mathbb{N}$, that $(x'_n + y'_n)$ is a $(\mathbf{w}, 2C)$-RIS, which by our assumption on (E_i) is also seminormalized in Y/X, and that there is a $\rho \in \mathbb{R}$ so that $\lim_{n\to\infty} \|T(x'_n + y'_n) - \rho(x'_n + y'_n)\|_{Y/X} = 0$.

But this implies that $\limsup_{n\to\infty} \|\lambda x'_n + \mu y'_n - \rho(x'_n + y'_n)\|_{Y/X} = 0$, and since (x'_n) and (y'_n) are seminormalized and $\max \operatorname{rg}(x'_n) < \min \operatorname{rg}(y'_n)$, for $n \in \mathbb{N}$, this yields, using our assumption on the FDD of X, that $\lambda = \mu = \rho$.

We claim now that $S = T - \lambda Id$ is a weakly compact operator, which, finishes our proof since by Schauder's theorem S is (weakly) compact if and only S^* is (weakly) compact and since by Schur's theorem all weakly compact operators on ℓ_1 are norm compact.

First note, that by what we just proved, using Propositions 2.8 and 2.11, it follows that the operator $\tilde{S} : Y \to Y/X$, $x \mapsto Q \circ S(x)$, where $Q : Y \to Y/X$ is the quotient mapping, is norm compact. Hence for a given $\varepsilon > 0$ there is an $N = N_\varepsilon$ so that $\operatorname{dist}(T(x), 2\|T\|B_X) < \varepsilon$, whenever $x \in \oplus_{j=N+1}^\infty F_j \cap B_Y$. Thus $T(B_Y) \subset W_\varepsilon + \varepsilon B_Y$, where $W_\varepsilon = 2MT\big(B_{\oplus_{j=1}^N F_j}\big) + 4M\|T\|B_X$. We thus showed that for every $\varepsilon > 0$ there is a relatively weakly compact set $W_\varepsilon \subset Y$ so that

$T(B_Y) \subset W_\varepsilon + \varepsilon B_Y$. We therefore deduce our claim from a well known characterization of weakly compactness (c.f. [Di]). ☐

3. Embedding background and other preliminaries

In this section we recall some of the embedding results established in [OS],[OS2], [FOSZ] and [FOS]. We first need to introduce some notation and terminology.

Let $\mathbf{E} = (E_i)_{i=1}^\infty$ be an FDD for a Banach space Z. $c_{00}(\oplus_{i=1}^\infty E_i)$ denotes the linear span of the E_i's and if $B \subseteq \mathbb{N}$, $c_{00}(\oplus_{i\in B} E_i)$ is the linear span of the E_i's for $i\in B$. $P_n = P_n^{\mathbf{E}} : Z \to E_n$ *is the n^{th} coordinate projection for the FDD*, i.e., $P_n(z) = z_n$ if $z = \sum_{i=1}^\infty z_i \in Z$ with $z_i \in E_i$, for all i. For a finite or cofinte set $A \subseteq \mathbb{N}$, we write $P_A = P_A^{\mathbf{E}} \equiv \sum_{n\in A} P_n^{\mathbf{E}}$. The *projection constant* of (E_n) in Z is

$$K = K(\mathbf{E}, Z) = \sup\left\{ \|P^{\mathbf{E}}_{[m,n]}\| : m \le n \right\} .$$

$\mathbf{E}$ is *bimonotone* if $K(\mathbf{E}, Z) = 1$.

The vector space $c_{00}(\oplus_{i=1}^\infty E_i^*)$, where E_i^* is the dual space of E_i, is naturally identified as a ω^*-dense subspace of Z^*. We write $Z^{(*)} = \overline{c_{00}(\oplus_{i=1}^\infty E_i^*)}^{Z^*}$. So $Z^{(*)} = Z^*$ if $(E_i)_{i=1}^\infty$ is shrinking, and then $\mathbf{E}^* = (E_i^*)_{i=1}^\infty$ is a boundedly complete FDD for Z^*.

As in the previous section we define for $z \in c_{00}(\oplus_{i=1}^\infty E_i)$ the *support of z* by $\mathrm{supp}_{\mathbf{E}}(z) = \{n : P_n^{\mathbf{E}}(z) \neq 0\}$. The *range of z*, is the smallest interval in $\mathbb{N}$ containing $\mathrm{supp}_{\mathbf{E}}(z)$ and we denote it by $\mathrm{rg}_{\mathbf{E}}(z)$.

A sequence $(z_i)_{i=1}^\ell$, where $\ell \in \mathbb{N}$ or $\ell = \infty$, in $c_{00}(\oplus_{i=1}^\infty E_i)$ is called a *block sequence* of (E_i) if $\max \mathrm{supp}_{\mathbf{E}}(z_n) < \min \mathrm{supp}_{\mathbf{E}}(z_{n+1})$ for all $n < \ell$.

Definition. [OSZ1] Let Z be a Banach space with an FDD $\mathbf{E} = (E_i)_{i=1}^\infty$. Let V be a Banach space with a normalized 1-unconditional basis $(v_i)_{i=1}^\infty$, and let $1 \le C < \infty$. We say that $(E_n)_{n=1}^\infty$ *satisfies subsequential C-V-upper estimates in Z*, if whenever $(z_i)_{i=1}^\infty$ is a normalized block sequence of $\mathbf{E}$ with $m_i = \min \mathrm{supp}_{\mathbf{E}}(z_i)$, $i \in \mathbb{N}$, then $(z_i)_{i=1}^\infty$ *is C-dominated by* $(v_{m_i})_{i=1}^\infty$. Precisely, for all $(a_i)_{i=1}^\infty \subseteq \mathbb{R}$,

$$\Big\| \sum_{i=1}^\infty a_i z_i \Big\| \le C \Big\| \sum_{i=1}^\infty a_i v_{m_i} \Big\| . \tag{18}$$

Similarly, $(E_n)_{n=1}^\infty$ *satisfies subsequential C-V-lower estimates* if every such $(z_i)_{i=1}^\infty$ C-dominates $(v_{m_i})_{i=1}^\infty$. We say that $(E_n)_{n=1}^\infty$ *satisfies subsequential V-upper estimates* or *subsequential V-lower estimates*, if for some $C > 1$ it satisfies C-V-upper estimates, respectively C-V-lower estimates.

In the case that (v_i) is a subsymmetric basis, i.e. if (v_i) is uniformly equivalent to all of its subsequences, we simply say that $(E_n)_{n=1}^\infty$ *satisfies C-V-upper estimates in Z* or $(E_n)_{n=1}^\infty$ *satisfies C-V-lower estimates in Z* if whenever $(z_i)_{i=1}^\infty$ is a normalized block then $(z_i)_{i=1}^\infty$ is C-dominated by $(v_i)_{i=1}^\infty$ or C-dominates $(v_i)_{i=1}^\infty$, respectively.

In the case that for some $C \ge 1$ and some $1 \le q \le p \le \infty$ (E_n) satisfies C-ℓ_p-lower and C-ℓ_q-upper estimates, we say that (E_n) *satisfies C-(p, q) estimates in Z.*

Definition. $[\mathbb{N}]^{<\omega}$ denotes the set of all finite subsets of $\mathbb{N}$ under the *pointwise topology*, i.e., the topology it inherits as a subset of $\{0,1\}^{\mathbb{N}}$ with the product topology. Let $\mathcal{A} \subseteq [\mathbb{N}]^{<\omega}$. We say $\mathcal{A}$ is

i) *compact* if it is compact in the pointwise topology,

ii) *hereditary* if for all $A \in \mathcal{A}$, if $B \subseteq A$ then $B \in \mathcal{A}$,

iii) *spreading* if for all $A = (a_1, \ldots, a_n) \in \mathcal{A}$ with $a_1 < a_2 < \cdots < a_n$ and all $B = (b_1, \ldots, b_n) \in [\mathbb{N}]^{<\omega}$ with $b_1 < b_2 < \cdots < b_n$ and $a_i \leq b_i$ for $i \leq n$, $B \in \mathcal{A}$. Such a B is called a *spread* of A,

iv) *regular* if $\{n\} \in \mathcal{A}$ for all $n \in \mathbb{N}$ and $\mathcal{A}$ is compact, hereditary and spreading.

Let $\mathcal{A} \subseteq [\mathbb{N}]^{<\omega}$ be a regular family. A sequence of sets in $[\mathbb{N}]^{<\omega}$, $A_1 < A_2 < \cdots < A_n$ (i.e., $\max A_i < \min A_{i+1}$ for $i < n$) is called *$\mathcal{A}$-admissible* if $(\min A_i)_{i=1}^n \in \mathcal{A}$.

Definition. Let $\mathcal{A} \subseteq [\mathbb{N}]^{<\omega}$ be a regular family of sets and let $0 < c < 1$. The Tsirelson space $T_{\mathcal{A},c}$ is the completion of c_{00} under the norm $\|\cdot\|_{\mathcal{A},c}$ which is given, implicitly, by the equation

$$\|x\|_{\mathcal{A},c} = \|x\|_\infty \vee \sup \Big\{ \sum_{i=1}^n c\|A_i x\|_{\mathcal{A},c} : n \in \mathbb{N} \ , \quad \text{and} \quad A_1 < \cdots < A_n \text{ is } \mathcal{A}\text{-admissible} \Big\}.$$

Here $A_i x = x|_{A_i}$. The unit vector basis of c_{00} is a normalized shrinking 1-unconditional basis for $T_{\mathcal{A},c}$. $T_{\mathcal{A},c}$ is reflexive (and contains no isomorph of c_0 or any ℓ_p).

If $\mathcal{A} = S_\alpha$ is the α^{th}-Schreier family of sets, where $\alpha < \omega_1$, we denote $T_{\mathcal{A},c}$ by $T_{\alpha,c}$. For more on these spaces (see e.g., [AT], [LTang], [OSZ2] and the references therein).

For $n \in \mathbb{N}$, $\mathcal{A}_n$ denotes the (regular) family of subsets of $\mathbb{N}$ with at most n elements. The following close connection between ℓ_p spaces and $T_{\mathcal{A}_n,c}$ was observed in [Be].

Proposition 3.1. [Be, Theorem 1.2]
If $c \leq \frac{1}{n}$ then $T_{\mathcal{A}_n,c}$ is naturally (i.e via the extension of the identity on c_{00}) isomorphic to c_0. If $c > \frac{1}{n}$ then $T_{\mathcal{A}_n,c}$ is naturally isomorphic to ℓ_p, where p is the solution of the equation

$$c = n^{-1+\frac{1}{p}}.$$

Our embedding theorems, 3.2 and 3.3 below, refer to the Szlenk index, $S_z(X)$, [Sz]. If X is separable then $S_z(X)$ is an ordinal with $S_z(X) < \omega_1$ if and only if X^* is separable. Also $S_z(T_{\alpha,c}) = \omega^{\alpha\cdot\omega}$ [OSZ2, Proposition 7]. If $S_z(X) < \omega_1$ then $S_z(X) = \omega^\beta$ for some $\beta < \omega_1$. Much has been written on the Szlenk index (e.g., see [AJO], [B2], [FOSZ], [G], [GKL], [JO], [L], [OSZ2]).

Theorem 3.2. [FOS, Theorem 1.3] *Let $\alpha < \omega_1$ and let X be a Banach space with separable dual. The following are equivalent.*

a) $S_z(X) \leq \omega^{\alpha\cdot\omega}$.

b) *X embeds into a Banach space Z having an* FDD *which satisfies subsequential $T_{\alpha,c}$-upper estimates for some $0 < c < 1$.*

Theorem 3.3. [OSZ2, Theorem A] *Let $\alpha < \omega_1$ and let X be a separable reflexive Banach space. The following are equivalent.*

a) *$S_z(X)\le\omega^{\alpha\cdot\omega}$ and $S_z(X^*)\le\omega^{\alpha\cdot\omega}$.*
b) *X embeds into a Banach space Z having an* FDD *which satisfies both subsequential $T_{\alpha,c}$-upper estimates and subsequential $T^*_{\alpha,c}$-lower estimates.*

The following notation was introduced in [FOS].

Definition. Let $\mathbf{E}=(E_i)_{i=1}^\infty$ be an FDD for a space X and let $0<c<1$. Let $x\in c_{00}(\oplus_{i=1}^\infty E_i)$. A block sequence of $\mathbf{E}$, $(x_1,\dots,x_\ell)$, is called a *c-decomposition of x* if

$$x=\sum_{i=1}^{\ell}x_i \quad\text{and, for every } i\le\ell,\text{ either }\ \#\mathrm{supp}_{\mathbf{E}}(x_i)=1\ \text{ or }\ \|x_i\|\le c\ . \tag{19}$$

Clearly every such x has a c-decomposition. The *optimal c-decomposition of x* is defined as follows. Set $n_1=\min\mathrm{supp}_{\mathbf{E}}(x)$ and assume $n_1<n_2<\dots<n_j$ have been defined. Let

$$n_{j+1}=\begin{cases} n_j+1\ , & \text{if } \|P^{\mathbf{E}}_{n_j}(x)\|>c\ ,\\ \min\{n:\|P^{\mathbf{E}}_{[n_j,n]}(x)\|>c\}\ , & \text{if } \|P^{\mathbf{E}}_{n_j}(x)\|\le c \text{ and the ``min'' exists,}\\ 1+\max\mathrm{supp}_{\mathbf{E}}(x)\ , & \text{otherwise.}\end{cases}$$

There will be a smallest ℓ so that $n_{\ell+1}=1+\max\mathrm{supp}_{\mathbf{E}}(x)$. We then set for $i\le\ell$, $x_i=P^{\mathbf{E}}_{[n_i,n_{i+1})}(x)$. Clearly $(x_i)_{i=1}^\ell$ is a c-decomposition of x. Moreover, if (E_i) is bimonotone and $j\le\lfloor\ell/2\rfloor$, then $\|x_{2j-1}+x_{2j}\|>c$.

Let $\mathcal{A}\subseteq[\mathbb{N}]^{<\omega}$ be regular. We say that the *FDD $(E_i)_{i=1}^\infty$ for X is $(c,\mathcal{A})$-admissible in X* if every $x\in S_X\cap c_{00}(\oplus_{i=1}^\infty E_i)$ has a c-decomposition, $(x_i)_{i=1}^k$, where $(\mathrm{supp}_{\mathbf{E}}(x_i))_1^\ell$ is *$\mathcal{A}$-admissible*, i.e., $(\min\mathrm{supp}_{\mathbf{E}}(x_i))_{i=1}^\ell\in\mathcal{A}$.

Proposition 3.4. (see [FOS, Proposition 3.1 and following remarks] and the remark below) *Let $\mathbf{E}=(E_i)_{i=1}^\infty$ be an* FDD *for a Banach space X whose projection constant is $b\ge1$. Let $\mathcal{A}\subseteq[\mathbb{N}]^{<\omega}$ be a regular family, let $0<c<1$ and $1\le C$.*

a) *Let $D\subseteq B_{X^{(*)}}\cap c_{00}(\oplus_{i=1}^\infty E_i^*)$ be d-norming for X for some $c<d/b^2\le1$. If every $x^*\in D$ has an $\mathcal{A}$-admissible c-decomposition (w.r.t. (E_i^*)), then $(E_i)_{i=1}^\infty$ satisfies subsequential c^{-1}-$T_{b^2c/d,\mathcal{B}}$-upper estimates where*

$$\mathcal{B}=\mathcal{B}_{\mathcal{A}}\equiv\Big\{\{n\}\cup B_1\cup B_2:n\in\mathbb{N},\ B_1,B_2\in\mathcal{A}\Big\}\cup\{\phi\}\ .$$

b) *Assume that $(E_i)_{i=1}^\infty$ satisfies subsequential C-$T_{\mathcal{A},c}$-upper estimates. Then there exists a regular family $\mathcal{G}=\mathcal{G}(c,C,\mathcal{A})$ such that $(E_i^*)_{i=1}^\infty$ is $(c,\mathcal{G})$-admissible in X^*. In fact, the optimal c-decomposition of every $x^*\in S_{X^*}\cap c_{00}(\oplus_{i=1}^\infty E_i^*)$ is $\mathcal{G}$-admissible.*

Moreover, if $\mathcal{A}=\mathcal{A}_k$, for some $k\in\mathbb{N}$, then there is a $\ell=\ell(k,b,c,C)$ so that in (b) we can choose $\mathcal{G}=\mathcal{A}_\ell$.

Remark. With the exception of the "moreover" statement, Proposition 3.4 coincides with Proposition 3.11 in [FOS] and the remark thereafter. But this additional observation follows immediately from the proof. Indeed, by first renorming X and changing C if necessary we can (as in the proof of [FOS, Proposition 3.11]) assume that (E_i) is bimonotone in X. Secondly, as the proof of part (b) of [FOS, Proposition 3.11] shows, the family $\mathcal{G}$, which satisfies the conditions in part (b) can be chosen to be $\mathcal{G} = \{n \cup A \cup B : A, B \in \mathcal{B}\}$, where

$$\mathcal{B} = \Big\{ B \in [\mathbb{N}]^{<\omega} : \Big\| \sum_{i\in B} t_i^* \Big\|_{T^*_{\mathcal{A}_k,c}} \leq C\, c^{-1} \Big\}$$

and where (t_i^*) denotes the unit vector basis of $T^*_{\mathcal{A}_k,c}$. Since (t_j^*) is 1-subsymmetric in $T^*_{\mathcal{A}_k,c}$, but not equivalent to the unit vector basis of c_0 we deduce the claim.

We now state a generalized version of Theorem A.

Theorem 3.5. *Assume that X is a separable reflexive space whose Szlenk index is ω_0. Then X embeds into a Bourgain-Delbaen space Z (as introduced in Section 2), whose dual is isomorphic to ℓ_1 and which has the property that all operators $T : Z \to Z$ are of the form $T = \lambda Id + K$, where λ is a scalar and K is compact operator on Z.*

Remarks. If X is a uniformly convex space, then, using the main result in [Ja] or [GG], one deduces that for some $1 < q \leq p < \infty$ the space X admits for some $C \geq 1$ C-(p, q)-tree estimates (see [OS, Definition 1.6] for the term *tree estimate* and Remark after [OS, Theorem 1.7]).

This implies (by the easy implication in) [OS2, Theorem 3] that the Szlenk index of X is ω_0. We therefore observe that Theorem 3.5 implies Theorem A.

Secondly, if X is reflexive and has Szlenk index ω_0, then, again by using (the non trivial part of) [OS2, Theorem 3], X satisfies (∞, q)-tree estimates for some $q > 1$, which yields by [OS2, Theorem 5] that X embeds into a reflexive space with an FDD (E_i) admitting C-(∞, q) estimates for some $C \geq 1$. Using also Proposition 3.1 we can therefore assume that for some m_0 and n_0 in $\mathbb{N}$, the space X in Theorem 3.5 has an FDD $\mathbf{E} = (E_i)$ which satisfies upper $T_{\mathcal{A}_{n_0},1/m_0}$ estimates. It also follows that we can increase n_0 if necessary, and that we can choose m_0 arbitrarily high, as long as we change n_0 accordingly.

4. Construction of the space Z in Theorem 3.5

According to the remarks after the statement of Theorem 3.5, and by renorming X, if necessary, we can assume that X has a bimonotone FDD (E_i), so that for some choice of m_0 and n_0 in $\mathbb{N}$ (E_i) satisfies $T_{\mathcal{A}_{n_0},1/m_0}$-upper estimates in X. We first apply Theorem A of [FOS], which allows us to embed X into a Bourgain-Delbaen space Y, whose dual is isomorphic to ℓ_1. Since we will want to exhibit certain properties of that space and we want to fit its construction into the framework introduced in Section 3, we need to recall this space in more detail.

Let $\varepsilon \in (0, 1)$ and $c \in (0, 1/16)$ be given, and choose $(\varepsilon_i)_{i=1}^\infty \subset (0, c)$ with $\varepsilon_i \downarrow 0$ so that

$$\sum_{i=1}^\infty \varepsilon_i < \frac{\varepsilon}{4} \quad \text{and} \quad \sum_{i>n} \varepsilon_i < \frac{\varepsilon_n}{2} \quad \text{for all} \quad n \in \mathbb{N}. \tag{20}$$

We choose $J_i \subseteq (0,1]$, $A_i \subseteq S_{E_i^*}$ and $R_i \subseteq (0,1)$ such that each is a finite set which is an $\varepsilon_i/2$-net for its respective superset. We shall assume $c - \varepsilon_i/2 \in R_i$ for each $i \in \mathbb{N}$. We write $\mathbf{E}^* = (E_i^*)_{i=1}^\infty$ for the FDD for X^*.

We next define $D \subseteq S_{X^*} \cap c_{00}(\oplus_{i=1}^\infty E_i^*)$ by

$$D = \left\{ \frac{\sum_{i\in B} a_i x_i^*}{\|\sum_{i\in B} a_i x_i^*\|} : \phi \neq B \in [\mathbb{N}]^{<\omega} \ , \ a_i \in J_i \ \text{ and } \ x_i^* \in A_i^* \ \text{ for } \ i \in B \right\} .$$

By Proposition 3.4 there is an $\ell_0 \in \mathbb{N}$ so that every $y^* \in D$ has a unique optimal c-decomposition, $(y_1^*, \dots, y_\ell^*)$, with $\ell \leq \ell_0$. We approximate this by elements of D and scalars r_i as follows. For $i \leq \ell$ choose $r_i \in R_{\min \operatorname{supp}_{\mathbf{E}^*}(y_i^*)}$ such that $|r_i - \|y_i^*\|\,| \leq \frac{1}{2}\varepsilon_{\min \operatorname{supp}_{\mathbf{E}^*}(y_i^*)}$, with $r_i \leq c$ if $\|y_i^*\| \leq c$. The latter is possible since $c - \frac{\varepsilon_j}{2} \in R_j$ for all j. Let $x_i^* = y_i^*/\|y_i^*\|$, which, by the definition of D, is again in D.

We then have

$$\|r_i x_i^* - y_i^*\| \leq \varepsilon_{\min \operatorname{supp}_{\mathbf{E}^*}(y_i^*)}/2 \ , \quad i \leq \ell \tag{21}$$

$$\Big\| y^* - \sum_{i=1}^{\ell} r_i x_i^* \Big\| \leq \sum_{i=1}^{\ell} \varepsilon_{\min \operatorname{supp}_{\mathbf{E}^*}(y_i^*)}/2 \text{ and} \tag{22}$$

$$\text{if } i \leq \ell \text{ and } \#\operatorname{supp}_{\mathbf{E}^*}(x_i^*) > 1 \ \text{ then } \ r_i \leq c \ . \tag{23}$$

We began with $y^* \in D$, took its optimal c-decomposition $(y_1^*, \dots, y_\ell^*)$ and approximated it by $(r_1 x_1^*, \dots, r_\ell x_\ell^*)$ where the x_i^*'s belong to D and were multiples of y_i^*. We call $(r_1 x_1^*, \dots, r_\ell x_\ell^*)$ the *D_c-decomposition of* $y^* \in D$ and fix such a choice for each y^*. If $x^* \in \bigcup_{j=1}^\infty A_j^*$, $x^* = (x^*)$ is its own D_c-decomposition.

Next we define $\tilde{\Gamma}$ (we use the superscript "$\sim$" because later we will replace $\tilde{\Gamma}$ by a set Γ which is defined by a sequence of Bourgain-Delbaen sets as introduced in Section 2) and a certain partial order on $\tilde{\Gamma}$ and use that to define the $\tilde{\Delta}_n$'s.

$$\tilde{\Gamma} = \left\{ (r_1 x_1^*, r_2 x_2^*, \dots, r_j x_j^*) : \begin{array}{c} j \geq 1 \text{ and there exists } y^* \in D \text{ so that } (r_1 x_1^*, \dots, r_j x_j^*) \text{ are} \\ \text{the first } j \text{ elements of the } D_c\text{-decomposition of } y^* \end{array} \right\} \cup \bigcup_{n=1}^\infty A_n^*.$$

We first define an order on the bounded intervals in $\mathbb{N}$ by $[n_1, n_2] < [m_1, m_2]$ if $n_2 < m_2$ or $n_2 = m_2$ and $n_1 > m_1$. It is not hard to see that this is a well ordering. It is instructive to list the first few elements in increasing order (we let $[n,n] = n$):

$$(I_n)_{n=1}^\infty = (1,\ 2,\ [1,2],\ 3,\ [2,3],\ [1,3],\ 4,\ [3,4],\ [2,4],\ [1,4],\ 5\ \dots)$$

We next define a subsequence $(k_j)_{j=1}^\infty$ of $\mathbb{N}$. If I_n consists of the singleton $\{j\}$ we put $k_j = n$. If $\tilde{\gamma} = (r_1 x_1^*, \dots, r_\ell x_\ell^*) \in \tilde{\Gamma}$ we let

$$\operatorname{rg}_{\mathbf{E}^*}\Big(\sum_{i=1}^{\ell} r_i x_i^*\Big) \equiv \operatorname{rg}_{\mathbf{E}^*}(\tilde{\gamma}) \quad \text{ and } \quad \operatorname{supp}_{\mathbf{E}^*}\Big(\sum_{i=1}^{\ell} r_i x_i^*\Big) \equiv \operatorname{supp}_{\mathbf{E}^*}(\tilde{\gamma}) \ .$$

We then define a partial order "$\leq$" on $\tilde{\Gamma}$ by $\tilde{\gamma}<\tilde{\eta}$ if either

- $\mathrm{rg}_{\mathbf{E}^*}(\tilde{\gamma})<\mathrm{rg}_{\mathbf{E}^*}(\tilde{\eta})$, or
- $\mathrm{rg}_{\mathbf{E}^*}(\tilde{\gamma})=\mathrm{rg}_{\mathbf{E}^*}(\tilde{\eta})$ has at least two elements, and $\tilde{\eta}$ is of length 1 while $\tilde{\gamma}$ is at least of length 2 (as finite sequences), or
- $\mathrm{rg}_{\mathbf{E}^*}(\tilde{\gamma})=\mathrm{rg}_{\mathbf{E}^*}(\tilde{\eta})=\{j\}$, for some $j\in\mathbb{N}$, $\tilde{\gamma}\in A_j^*$, and $\tilde{\eta}=(rx^*)$, for some $r\in I_j$ and $x^*\in A_j^*$.

This yields a rank of $\tilde{\gamma}$, denoted by $\mathrm{rk}(\tilde{\gamma})\in\tilde{\Gamma}$, for $\tilde{\gamma}\in\tilde{\Gamma}$: if $\tilde{\gamma}$ is minimal in $\tilde{\Gamma}$ with respect to $\leq$ we put $\mathrm{rk}(\tilde{\gamma})=1$, and, assuming we defined $\tilde{\Gamma}_n=\{\tilde{\gamma}:\mathrm{rk}(\tilde{\gamma})\leq n\}$, we assign to all minimal elements of $\tilde{\Gamma}\setminus\tilde{\Gamma}_n$ the rank $n+1$. It is easy to see that for $\tilde{\gamma}\in\tilde{\Gamma}$

$$\mathrm{rk}(\tilde{(\gamma)})=\begin{cases}2n-1 & \text{if } \mathrm{rg}_{E^*}(\tilde{\gamma})=I_n \text{ and either } n=k_j,\text{ for some } j\in\mathbb{N},\text{ with } \tilde{\gamma}\in A_j^*\\ & \qquad\qquad \text{or length}(\tilde{\gamma})\geq 2,\\ 2n & \text{if } \mathrm{rg}_{E^*}(\tilde{\gamma})=I_n \text{ and length}(\tilde{\gamma})=1 \text{ and } \tilde{\gamma}\notin\bigcup_{j\in\mathbb{N}}A_j^*.\end{cases}$$

Thus, the elements of A_1^* (which are their own D_c-decomposition) have rank 1. All elements of the form $r\cdot x^*$, with $r\in R_1\subset(0,1)$ and $x^*\in A_1^*$, which are the beginning of some D_c-decomposition of an element in D have rank 2. All elements of A_2^* have rank 3 and so on. We set for $n\in\mathbb{N}$, $\tilde{\Delta}_n=\{\tilde{\gamma}\in\tilde{\Gamma}:\mathrm{rk}(\tilde{\gamma})=n\}$.

We now define Δ_1, and $\Delta_n^{(0)}$ and $\Delta_n^{(1)}$, for $n>1$, satisfying the conditions in Definition 2.1. $\Delta_n=\Delta_n^{(0)}\cup\Delta_n^{(1)}$ will be, in a natural way, a bijective image of $\tilde{\Delta}_n$. Put $\Delta_1=A_1^*$, and assume we have defined for some $n\in\mathbb{N}$, Δ_1, $\Delta_j^{(0)}$ and $\Delta_j^{(1)}$, if $1<j\leq 2n-1$, together with a bijection $\tilde{(\cdot)}:\Gamma_{2n-1}\to\tilde{\Gamma}_{2n-1}$, which carries $\tilde{\Delta}_j$ into Δ_j, $j\leq n$, and where $\Gamma_{2n-1}=\bigcup_{j\leq 2n-1}\Delta_j=\bigcup_{j\leq 2n-1}(\Delta_j^{(0)}\cup\Delta_j^{(1)})$, and $\tilde{\Gamma}_{2n-1}=\{\tilde{\gamma}\in\tilde{\Gamma},\mathrm{rk}(\tilde{\gamma})\leq 2n-1\}$.

We then consider the following four cases:

Case 1. If $\tilde{\gamma}=(r_1x_1^*)\in\tilde{\Delta}_{2n}$ and $x_1^*\in A_j^*$ for some $j\in\mathbb{N}$, then it follows that $r_1<1$ and $n=k_j$ and we put $\gamma=(2n,cr_1,e_\eta^*,0)$, where $\tilde{\eta}=x^*\in\tilde{\Delta}_{2n-1}=A_j^*$. We let therefore in this case $\Delta_{2n}^{(0)}=\{\gamma:\tilde{\gamma}\in\tilde{\Delta}_{2n}\}$ and $\Delta_{2n}^{(1)}=\emptyset$.

Case 2. If $\tilde{\gamma}=(r_1x_1^*)\in\tilde{\Delta}_{2n}$ and $x_1^*\notin\bigcup_{j\in\mathbb{N}}\mathrm{span}(A_j^*)$, then $n\notin\{k_j:j\in\mathbb{N}\}$ and $r_1\leq c$, and we put $\gamma=(2n,r_1,e_\eta^*,0)$ where $\tilde{\eta}$ is the D_c decomposition of x_1^*. In that case we put $\Delta_{2n}^{(0)}=\{\gamma:\tilde{\gamma}\in\tilde{\Delta}_n\}$ and $\Delta_{2n}^{(1)}=\emptyset$.

Case 3. If $\tilde{\gamma}=x^*\in A_j^*=\tilde{\Delta}_{2n+1}$, for some $j\in\mathbb{N}$, and thus $n=k_j-1$, for some $j\in\mathbb{N}$, then set $\gamma=(2n+1,c,0,x^*)$. In the notation of Definition 2.1, $\beta=c$, $b^*=0$, and $V_{2n+1,1,0}=A_j^*$. Set $\Delta_{2n+1}^{(0)}=\{\gamma:\tilde{\gamma}\in\Delta_{2n+1}\}$ and $\Delta_{2n+1}^{(1)}=\emptyset$.

Case 4. If $\tilde{\gamma}=(r_1x_1^*,r_2x_2^*,\ldots,r_\ell x_\ell^*)\in\tilde{\Delta}_{2n+1}$, $n\notin\{k_j-1:j\in\mathbb{N}\}$, then $\ell>1$ and we let $\gamma=(2n+1,1,k,\xi,c_\ell r_\ell,e_\eta^*,0)$ where $\tilde{\xi}=(r_1x_1^*,r_2x_2^*,\ldots,r_{\ell-1}x_{\ell-1}^*)$, $k=\mathrm{rk}(\tilde{\xi})$, $\tilde{\eta}$ is the

D_c decomposition of x^*_ℓ, and $c_\ell = \begin{Bmatrix} c & \text{if } x^*_\ell \in \cup A^*_j \\ 1 & \text{otherwise} \end{Bmatrix}$. Set $\Delta^{(1)}_{2n+1} = \{\gamma : \tilde{\gamma} \in \tilde{\Delta}_{2n+1}\}$ and $\Delta^{(0)}_{2n+1} = \emptyset$.

It follows that $\mathrm{wt}(\gamma) \le c$ in all cases. The remarks after Proposition 2.3 gives us that the corresponding Bourgain Delaben space Y is an $\mathcal{L}_{2,\infty}$-space which has an FFD (F_i), with (we define $\mathrm{rk}(\gamma) = \mathrm{rk}(\tilde{\gamma})$, for $\gamma \in \Gamma$) $F_i = \mathrm{span}(d_\gamma : \mathrm{rk}(\gamma) = i)$, whose decomposition constant M is not larger than 2.

Secondly, consider for $i \in \mathbb{N}$ the map

$$\phi_i : E_i \to F_{2k_i-1}, \qquad x \mapsto c^{-1} \sum_{x^* \in A_i} x^*(x) d_{(2k_i-1,1,0,x^*)}$$

(recall that $\Delta_{2k_i-1} = \Delta^{(0)}_{2k_i-1} \equiv \tilde{\Delta}_{2k_i-1} = A^*_i$, for all $i \in \mathbb{N}$). Then, similar to Proposition 4.5 in [FOS], the map

$$\phi : c_{00}(\oplus E_i) \to : c_{00}(\oplus F_j), \qquad \sum_{i=1} x_i \mapsto \sum_{i=1} \phi_i(x_i)$$

extends to an isomorphic embedding, which we still denote by ϕ of X into Y, with

$$(1+\varepsilon)^{-1}\|x\| \le \|\phi(x)\| \le (1+\varepsilon)\|x\|, \text{ for } x \in X. \tag{24}$$

Thirdly, we note that for $\gamma \in \Gamma$ the age of γ equals to the length of $\tilde{\gamma}$ which is not larger than ℓ_0. Using the analysis (14) of γ we can write e^*_γ as

$$e^*_\gamma = \sum_{j=1}^{a} d^*_{\xi_j} + \beta_j P^*_{(p_{j-1},p_j)}(b^*_j) = \sum_{j=1}^{a} d^*_{\xi_j} + \beta_j b^*_j. \tag{25}$$

We note that as in [FOS], we conveniently have that $P^*_{(p_{j-1},p_j)}(b^*_j) = b^*_j$ for all $1 \le j \le a$. We further note that $\beta_j \le c \le 1/16$ and thus $\|\beta_j b^*_j\| \le 1/16$. From the analysis (25) of γ it follows therefore that there is a c-decomposition of e^*_γ which has no more than $2\ell_0$ elements, thus Proposition 3.4 yields that (F_i) is a shrinking FDD of Y, which for some $c' \in (0,1)$ and $1 \le C' < \infty$ has C'-$T_{\mathcal{A}_{6\ell_0,c'}}$-upper estimates.

We make one last formal change of the elements of $\Delta^{(0)}_n$ and $\Delta^{(1)}_n$ and change their elements (n,β,b^*,v) and $(n,\alpha,k,\xi,\beta,b^*,v)$ in case that $\beta \le c$ into $(n,c,\frac{\beta}{c}b^*,(\beta,v))$ and $(n,\alpha,k,\xi,\frac{\beta}{c}b^*,(\beta,v))$. As noted in the Remark after Definition 2.5 this will not change the family $(c^*_\gamma)_{\gamma\in\Gamma}$ and, thus, neither the space Y. But the new sequence of Bourgain-Delbaen sets, which we still denote by (Δ_n), has comparable weights, and

$$q_0 = \inf_{\gamma\in\Gamma} \frac{\overline{\mathrm{w}}(\gamma)}{\underline{\mathrm{w}}(\gamma)} = 1.$$

In our next step we will use the construction in [AH], and increase the sets Δ_n to sets $\overline{\Delta}_n = \Delta_n \cup \Theta_n$ in such a way, that X will still embed into a Bourgain-Delbaen space Z corresponding to the Bourgain-Delbaen sets $(\overline{\Delta}_n)$, and will have the additional property that all operators on it will be compact perturbations of a scalar multiple of the identity.

Our construction so far has dealt with a general constant c satisfying $0 < c < \frac{1}{16}$. We now impose the further condition that $\frac{1}{c} \in \mathbb{N}$. This condition will be convenient for us now as we introduce some modifications based on [AH]. As in [AH] we start with two sequences (m_j) and (n_j) in $\mathbb{N}$ satisfying the following properties.

$$(26) \qquad \sqrt{n_1/2} > {}^1\!/c = m_1 > 16, \quad m_{j+1} \geq m_j^2, \quad n_{j+1} \geq (16n_j)^{\log_2(m_{j+1})}, \quad \text{and } n_1 \geq 2\ell_0.$$

The FDD (F_i) which we have constructed satisfies C_q-ℓ_q-upper estimates for some $1 < q < \infty$ and $C_q \geq 1$. Based on the values of q and C_q, we will impose the following additional property on (m_j) and (n_j).

$$(27) \qquad C_q m_j^3 \leq n_j^{1-1/q}.$$

By induction we define for every $n \in \mathbb{N}$ sets $\Theta_n^{(0)}$ and $\Theta_n^{(1)}$. In the notation of Definition 2.1 we will always have $\alpha = 1$, and the set of all free variables will always be a singleton, we will therefore suppress this dependency, and write (n, β, b^*) for elements in $\Theta_n^{(0)}$ and (n, k, ξ, β, b^*) for elements in $\Theta_n^{(1)}$.

We let $\Theta_1^{(0)} = \Theta_1^{(1)} = \emptyset$, and assuming we defined $\Theta_j^{(0)}$ and $\Theta_j^{(1)}$, for $j = 1, 2 \ldots, n$, we let $\overline{\Delta}_j^{(0)} = \Delta_j^{(0)} \cup \Theta_j^{(0)}$, $\overline{\Delta}_j^{(1)} = \Delta_j^{(1)} \cup \Theta_j^{(1)}$, $\Theta_j = \Theta_j^{(0)} \cup \Theta_j^{(1)}$, $\overline{\Delta}_j = \overline{\Delta}_j^{(0)} \cup \overline{\Delta}_j^{(1)}$, $\Lambda_j = \bigcup_{i=1}^j \Theta_j$ and $\overline{\Gamma}_j = \bigcup_{i=1}^j \overline{\Delta}_i$, for $j = 1, \ldots n$. We also assume that, so far $(\overline{\Delta}_j)_{j=1}^n$ satisfies the conditions of Bourgain-Delbaen sets in Definition 2.1. The terms rank, type, weight, free variable, analysis and age of γ are therefore defined for all $\gamma \in \overline{\Gamma}_n$. Also the functionals $c_\gamma^* \in \ell_1(\overline{\Gamma}_n)$, $\gamma \in \overline{\Gamma}$, as well as the projections $P_{[p,q]}^*$ (on $\ell(\overline{\Gamma}_n)$), $1 \leq p \leq n$, are defined. We also assume that, so far, $\sigma : \overline{\Gamma}_n \to \{1, 2 \ldots, \#\overline{\Gamma}_n\}$ is a bijection with $\sigma(\gamma) \in (\#\overline{\Gamma}_{j-1}, \#\overline{\Gamma}_j]$, if $j \in \overline{\Delta}_j$, for $j \leq n$.

Also note that for $i \in \mathbb{N}$, so that $k_i \leq n$, it follows that $E_i \subset F_{k_i}$. , and we have the following (natural) embeddings

$$\oplus_{i,k_i \leq n} E_i \hookrightarrow \oplus_{j \leq n} F_j \equiv_M \ell_\infty(\Gamma_n)$$

We will identify $X \subset \ell_\infty(\Gamma)$ with $X \oplus 0 \subset \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda) = \ell_\infty(\overline{\Gamma})$. Spefically, if $b^* \in \ell_1(\overline{\Gamma}_n)$, we denote by $b^*|_X$ the functional defined by the restriction of b^* onto the space $\oplus_{i,k_i \leq n} E_i \oplus 0 \subset \ell_\infty(\Gamma_n) \oplus \ell_\infty(\Lambda_n) = \ell_\infty(\overline{\Gamma}_n)$.

For $0 \leq p \leq n$, we choose finite sets $\overline{B}_{(p,n]}^*$ from the set

$$\left\{ b^* \in \oplus_{i=p+1}^n \bar{F}_i^* : \|b^*\|_{\ell_1} \leq 1, b^*|_X \equiv 0 \right\},$$

such that $\overline{B}_{(p,n]}^*$ is ε_n-dense in the $\ell_1(\Gamma_n)$ norm. We assume, without loss of generality, that $\overline{B}_{(p,n]}^* \subset \overline{B}_{(q,m]}^*$ if $q \leq p < n \leq m$. We will also require that if $\gamma \in \overline{\Gamma} \setminus \Gamma$ and $e_\gamma^* \in \oplus_{i=p+1}^n \bar{F}_i^*$ then $e_\gamma^* \in \overline{B}_{(p,n]}^*$. Note that the conditions that $e_\gamma^*|_X = 0$ and $\|e_\gamma^*\|_{\ell_1(\Gamma)} \leq 1$ are automatically satisfied for $\gamma \in \overline{\Gamma} \setminus \Gamma$. For $f^* \in c_{00}(F_i)$ we define the range of f^* to be the smallest interval $range(f^*) = [p, n]$ such that $f^* \in \oplus_{i=p}^n \bar{F}_i^*$. If we have constructed $\overline{\Gamma}_n$, then the Bourgain-Delbaen techniques produce the FDD $(\bar{F}_i^*)_{i=1}^n$. Thus we can incorporate the FDD $(\bar{F}_i^*)_{i=1}^n$ into the creation of Θ_{n+1}.

The sets $\Theta_n^{(0)}$ and $\Theta_n^{(1)}$ are now defined as follows:

(28)
$$\Theta_{n+1}^{(0)} = \begin{array}{l} \bigcup_{p=0}^{n-1} \bigcup_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \left\{ \left(n+1, p, \frac{1}{m_{2j}}, b^*\right) : b^* \in \overline{B}^*_{(p,n]} \right\} \\ \cup \bigcup_{p=0}^{n-1} \bigcup_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} \left\{ \left(n+1, p, \frac{1}{m_{2j-1}}, e^*_\eta\right) : \begin{array}{c} \eta \in \Lambda_n, \min \mathrm{range}(e^*_\eta) > p, \\ \mathrm{wt}(\eta) = \frac{1}{m_{4i-2}}, \text{ with } m_{4i-2} > n^2_{2j-1} \end{array} \right\} \end{array}$$

and

(29)
$$\Theta_{n+1}^{(1)} = \begin{array}{l} \bigcup_{p=2}^{n-1} \bigcup_{j=1}^{\lfloor \frac{p}{2} \rfloor} \left\{ \left(n+1, p, e^*_\xi, \frac{1}{m_{2j}}, b^*\right) : \xi \in \Theta_p, \mathrm{wt}(\xi) = \frac{1}{m_{2j}}, \mathrm{age}(\xi) < n_{2j}, b^* \in \overline{B}^*_{(p,n]} \right\} \\ \cup \bigcup_{p=2}^{n-1} \bigcup_{j=1}^{\lfloor \frac{p+1}{2} \rfloor} \left\{ \left(n+1, p, e^*_\xi, \frac{1}{m_{2j-1}}, e^*_\eta\right) : \begin{array}{c} \xi \in \Theta_p, \mathrm{wt}(\xi) = \frac{1}{m_{2j-1}}, \mathrm{age}(\xi) < n_{2j-1}, \\ \eta \in \Lambda_n, \min \mathrm{range}(e^*_\eta) > p, \mathrm{wt}(\eta) = \frac{1}{m_{4\sigma(\xi)}} \end{array} \right\} \end{array}$$

The sets $\overline{\Delta}_n = \Delta_n \cup \Theta_n$ form Bourgain-Delbaen sets as in Definition 2.1. We have dropped the variables α and f from Definition 2.1 as for $\gamma \in \Theta_n$ we always set $\alpha, f \equiv 1$. The remark after Proposition 2.3 holds for our new sets $\overline{\Delta}_n$, and thus we have that $(\bar{F}^*_n)$ is an FDD for $\ell_1(\bar{\Gamma})$ with decomposition constant not larger than 2. We denote Z to be the Bourgain-Delbaen space associated to $(\Delta_n : n \in \mathbb{N})$, which again by the remark after Proposition 2.3 is a $\mathcal{L}_{2,\infty}$ space.

In our construction of $\overline{B}^*_{(p,n]}$ we required that $e^*_\gamma \in \overline{B}^*_{(p,n]}$ if $\gamma \in \overline{\Gamma} \setminus \Gamma$ and $range(e^*_\gamma) \subset (p, n]$. In some circumstances, this allows us to conveniently combine all four possible cases for $\gamma \in \overline{\Gamma} \setminus \Gamma$ into one general case. For instance, if $\gamma \in \overline{\Gamma} \setminus \Gamma$ has age a and weight m_j^{-1}, then the evaluation analysis of γ is given by

$$e^*_\gamma = \sum_{i=1}^{a} d^*_{\xi_i} + m_j^{-1} \sum_{i=1}^{a} b^*_i,$$

where $b^*_i \in \overline{B}_{(p_{i-1},p_i]}$ and $\xi_i \in \Theta_{p_i}$ for some sequence of non-negative integers $(p_i)_{i=0}^a \subset \mathbb{N}_0$. It is important to point out that we denote include the projection operators $P^*_{(p_{i-1},p_i)}$ as we have guaranteed that $\min range(b^*_i) > p_{i-1}$ and hence $P^*_{(p_{i-1},\infty)} b^*_i = b^*_i$.

Our first goal is to show that X is naturally isomorphic to a subspace of Z. We are given that $X \subset Y \subset \ell_\infty(\Gamma)$ and that $Z \subset \ell_\infty(\Gamma \cup \Lambda) = \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda)$. We will prove that the set Λ has been defined in such a way so that $X \oplus 0 \subset Z \subset \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda)$, and hence X naturally embeds as a subspace of Z. Before proving this property, we will need the following lemma. Recall that we identify $X \subset \ell_\infty(\Gamma)$ with $X \oplus 0 \subset \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda) = \ell_\infty(\overline{\Gamma})$.

Lemma 4.1. *If $\gamma \in \overline{\Gamma} \setminus \Gamma$ then $e^*_\gamma|_X = c^*_\gamma|_X = d^*_\gamma|_X = 0$.*

Proof. We are identifying $X \subset \ell_\infty(\Gamma)$ with $X \oplus 0 \subset \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda) = \ell_\infty(\overline{\Gamma})$, which means that $e^*_\gamma|_X = 0$ for all $\gamma \in \overline{\Gamma} \setminus \Gamma$. We have that $e^*_\gamma = d^*_\gamma + c^*_\gamma$, and thus it will be sufficient for us to just prove that $c^*_\gamma|_X = 0$.

There are two possible cases for $\gamma \in \overline{\Gamma} \setminus \Gamma$. In the first case $\gamma = (n+1, p, m_j^{-1}, b^*) \in \Theta_{n+1}^{(0)}$ for some $b^* \in \overline{B}^*_{(p,n]}$. Thus $c^*_\gamma|_X = m_j^{-1} b^*|_X = 0$ as $b^*|_X = 0$ for all $b^* \in \overline{B}^*_{(p,n]}$.

In the second case, $\gamma = (n+1, p, e^*_\xi, m_j^{-1}, b^*) \in \Theta^{(1)}_{n+1}$ for some $b^* \in \overline{B}^*_{(p,n]}$ and $\xi \in \Theta_p$. Thus $c^*_\gamma|_X = e^*_\xi|_X + m_{2j}^{-1} b^*|_X = 0$ as $e^*_\xi|_X = 0$ for all $\xi \in \overline{\Gamma} \setminus \Gamma$ and $b^*|_X = 0$ for all $b^* \in \overline{B}^*_{(p,n]}$. ☐

Theorem 4.2. *The space $X \oplus 0 \subset \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda)$ is a subspace of $Z \subset \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda) = \ell_\infty(\overline{\Gamma})$.*

Proof. We will first recall some of the technicalities of the Bourgain-Delbaen construction. We have extension operators $J_n : \ell_\infty(\Gamma_n) :\to \ell_\infty(\Gamma)$ such that for all $y \in \ell_\infty(\Gamma_n)$,

$$J_n y(\gamma) = \begin{cases} e^*_\gamma(y) & \text{if } \gamma \in \Gamma_n; \\ c^*_\gamma(J_n y) & \text{if } \gamma \in \Gamma \setminus \Gamma_n. \end{cases}$$

We denote $Y_n = J_n(\ell_\infty(\Gamma_n))$. The $\mathcal{L}_\infty$ space $Y \subset \ell_\infty(\Gamma)$ is then defined by $Y = \overline{\cup Y_n}$.

Modifying our Bourgan-Delbaen space Y into the space Z gives new extension operators $\bar{J}_n : \ell_\infty(\overline{\Gamma}_n) :\to \ell_\infty(\overline{\Gamma})$ such that for all $y \in \ell_\infty(\overline{\Gamma}_n)$,

$$\bar{J}_n y(\gamma) = \begin{cases} J_n y|_{\ell_\infty(\Gamma_n)}(\gamma) & \text{if } \gamma \in \Gamma; \\ e^*_\gamma(y) & \text{if } \gamma \in \overline{\Gamma}_n; \\ c^*_\gamma(\bar{J}_n y) & \text{if } \gamma \in \overline{\Gamma} \setminus \overline{\Gamma}_n. \end{cases}$$

We denote $Z_n = \bar{J}_n(\ell_\infty(\overline{\Gamma}_n))$. The $\mathcal{L}_\infty$ space $Z \subset \ell_\infty(\overline{\Gamma})$ is then defined by $Z = \overline{\cup Z_n}$.

Recall that (E_i) is the FDD for X, and that (F_i) is the FDD for Y defined by $F_i = J_i(\ell_\infty(\Delta_i))$. Our specific construction of Y gave that $E_i \subset F_{k_i}$ for some sequence $(k_i) \in [\mathbb{N}]^\omega$. Thus if $x \in \oplus_{i=1}^n E_i$, then $x = J_{k_n} R_{k_n} x$ where $R_{k_n} : \ell_{\infty(\Gamma)} \to \ell_\infty(\Gamma_{k_n})$ is the restriction operator. Our goal is to now show that $(x, 0) = \bar{J}_{k_n}(R_{k_n} x, 0) \in \ell_\infty(\Gamma) \oplus \ell_\infty(\Lambda)$.

First, we check for $\gamma \in \Gamma$ that

$$\bar{J}_{k_n}(R_{k_n} x, 0)(\gamma) = J_{k_n} R_{k_n} x(\gamma) = x(\gamma).$$

We assume for a fixed $N \in \mathbb{N}$ that $\bar{J}_{k_n}(R_{k_n} x, 0)(\gamma) = (x, 0)(\gamma) = 0$ for all $\gamma \in \Lambda_n$. If $\gamma \in \Theta_{N+1}$ then $c^*_\gamma \in \ell_\infty(\overline{\Gamma}_N)$ and by Lemma 4.1 we have that $c^*_\gamma|_X = 0$. We now have the following,

$$\begin{aligned} \bar{J}_{k_n}(R_{k_n} x, 0)(\gamma) =& c^*_\gamma(J_{k_n}(R_{k_n} x, 0)) \\ &= c^*_\gamma(x, 0) \quad \text{as } c^*_\gamma \in \ell_\infty(\overline{\Gamma}_N) \\ &= 0 \quad \text{as } c^*_\gamma|_X = 0 \end{aligned}$$

Thus by induction, we have that $\bar{J}_{k_n}(R_{k_n} x, 0)(\gamma) = (x, 0)(\gamma)$ for all $\gamma \in \overline{\Gamma}$. Hence, $(x, 0) = \bar{J}_{k_n}(R_{k_n} x, 0) \in Z$. Which implies our desired result that $X \oplus 0 \subset Z$. ☐

Proposition 4.3. *Let $\gamma, \gamma' \in \Theta^{(1)}_{n+1}$ each have the same odd weight $\frac{1}{m_{2j-1}}$ and have respective analyses $(p_i, e^*_{\eta_i}, \xi_i)_{1 \le i \le a}$ and $(p'_i, e^*_{\eta'_i}, \xi'_i)_{1 \le i \le a'}$. If $a \ge a'$ then there exists $1 \le \ell \le a$ such that $\xi'_i = \xi_i$ for all $i < \ell$ and $wt(\eta_j) \ne wt(\eta'_i)$ for all j and all $\ell < i \le a'$*

Proof. We choose $1 \le \ell \le a'$ to be maximal such that $\xi'_i = \xi_i$ for all $i < \ell$. If $\ell = a'$ then the proposition holds. If $\ell < a'$ it must be that $\xi'_\ell \ne \xi_\ell$, and hence $wt(\eta_\ell) = m_{4\sigma(\xi_\ell)} \ne m_{4\sigma(\xi'_\ell)} = wt(\eta'_\ell)$. In this set up, ages are simply given by $age(\xi_i) = i$ and $age(\xi'_j) = j$ for all $1 \le j \le a$ and $1 \le i \le a'$. Thus whenever $i \ne j$ we have that $wt(\eta_j) = m_{4\sigma(\xi_j)} \ne m_{4\sigma(\xi'_i)} = wt(\eta'_i)$. If

$j > \ell$ then the analysis of ξ_j is $(p_i, e^*_{\eta_i}, \xi_i)_{1\le i\le j-1}$ and the analysis of ξ'_j is $(p'_i, e^*_{\eta'_i}, \xi'_i)_{1\le i\le j-1}$. The elements ξ_j and ξ'_j clearly have different analyses as $\xi'_\ell \neq \xi_\ell$, and thus $\xi_j \neq \xi'_j$. We then have that $wt(\eta_j) = m_{4\sigma(\xi_j)} \neq m_{4\sigma(\xi'_j)} = wt(\eta'_j)$. We have covered all the cases, and thus the proposition is proven. $\square$

If we are given some $\gamma \in \overline{\Gamma}$ then we can find the analysis of γ through simple iteratation. Conversely, it will be important for us to be able to choose an element $\gamma \in \overline{\Gamma}$ which has some specified analysis. The following lemmas state essentially that if we satisfy some important conditions, then we are able to choose such a γ.

Lemma 4.4. *Let a, j be positive integers such that $a \le n_{2j}$. If $p_0 < p_1 < p_2 < \cdots < p_a$ are natural numbers with $2j \le p_1$ and b^*_r is a functional in $\overline{B}^*_{(p_{r-1},p_r-1]}$ for all $1 \le r \le a$, then there are elements $\xi_r \in \Theta_{p_r}$ each with weight $\frac{1}{m_{2j}}$ such that the analysis of $\gamma = \xi_a$ is $(p_r, b^*_r, \xi_r)^a_{r=1}$.*

Proof. It is specified that $2j \le p_1$, and thus $(p_1, p_0, \frac{1}{m_{2j}}, b^*_1) \in \Theta^{(0)}_{p_1}$. We now assume that $1 \le k < a$ and that ξ_k has been found with analysis $(p_r, b^*_r, \xi_r)^k_{r=1}$ and weight $\frac{1}{m_{2j}}$. We have that $age(\xi_k) = k < a \le n_{2j}$, and thus there exists $\xi_{k+1} = (p_{k+1}, p_k, e^*_{\xi_k}, \frac{1}{m_{2j}}, b^*_{k+1}) \in \Theta^{(1)}_{p_{k+1}}$. Thus $e^*_{\xi_{k+1}} = d^*_{\xi_{k+1}} + e^*_{\xi_k} + \frac{1}{m_{2j}} b^*_{k+1}$, and hence the analysis of ξ_{k+1} is $(p_r, b^*_r, \xi_r)^{k+1}_{r=1}$. The proof is then complete by induction. $\square$

Lemma 4.5. *Let a, j_0 be positive integers such that $a \le n_{2j_0-1}$. Let $p_0 < p_1 < p_2 < \cdots < p_a$ be natural numbers with $2j_0 - 1 \le p_1$. Let $(\eta_r)^a_{r=1} \subset \Lambda$ with $range(e^*_{\eta_r}) \subset (p_{r-1}, p_r]$ such that $wt(\eta_1) = \frac{1}{m_{4j_1-2}}$ for some $j_1 \in \mathbb{N}$ with $m_{4j_1-2} > n^2_{2j_0-1}$ and $wt(\eta_r) = \frac{1}{m_{4\sigma(\eta_{r-1})}}$ for all $2 \le r \le a$. Then there exist elements $\xi_r \in \Theta_{p_r}$ each with weight $\frac{1}{m_{2j_0-1}}$ such that the analysis of $\gamma = \xi_a$ is $(p_r, e^*_{\eta_r}, \xi_r)^a_{r=1}$.*

Proof. We have that $2j_0 - 1 \le p_1$ as well as $wt(\eta_1) = m^{-1}_{4j_1-2} < n^{-2}_{2j_0-1}$ and $range(e^*_{\eta_1}) \subset (p_0, p_1)$, which implies that $(p_1, p_0, \frac{1}{m_{2j_0-1}}, e^*_{\eta_1}) \in \Theta^{(0)}_{p_1}$. We now assume that $1 \le k < a$ and that ξ_k has been found with analysis $(p_r, e^*_{\eta_r}, \xi_r)^k_{r=1}$ and weight $\frac{1}{m_{2j_0-1}}$. We have that $age(\xi_k) = k < a \le n_{2j_0-1}$ as well as $wt(\eta_{k+1}) = \frac{1}{m_{4\sigma(\eta_k)}}$, and $range(e^*_{\eta_r}) \subset (p_{r-1}, p_r]$. Thus there exists $\xi_{k+1} = (p_{k+1}, p_k, e^*_{\xi_k}, \frac{1}{m_{2j_0-1}}, e^*_{\eta_{k+1}}) \in \Theta^{(1)}_{p_{k+1}}$. Hence the anlaysis of ξ_{k+1} is $(p_r, e^*_{\eta_r}, \xi_r)^{k+1}_{r=1}$. The proof is then complete by induction. $\square$

Lemma 4.6. *If $p < q$ and $x \in \oplus^q_{i=p+1} F_i$ such that $\|x\|_{Z/X} = 1$ then there exists $b^* \in \overline{B}^*_{(p,q]}$ with $b^*(x) > \frac{1}{8} - \varepsilon_q$.*

Proof. As $\|x\|_{Z/X} = 1$, there exists $x^* \in S_{[\oplus^\infty_{i=1} F_i]^*}$ such that $x^*(x) = 1$ and $x^*|_X = 0$. We then set $b^*_0 = \frac{1}{8} P^*_{[p+1,q]} x^*$. We have that $\|b^*_0\|_{\ell_1(\Gamma_q)} \le 2\|b^*_0\| \le \frac{1}{4}\|P^*_{[p+1,q]}\|\|x^*\| \le 1$. If $x_0 \in X$, then our particular embedding of X into Z results in $P_{[p+1,q]} x_0 \in X$. Thus $b^*_0(x_0) = \frac{1}{8} x^*(P_{[p+1,q]} x) = 0$, as $x^*|_X = 0$. Combining these properties gives that $b^*_0 \in \left\{ b^* \in \oplus^n_{i=p+1} \bar{F}^*_i : \|b^*\|_{\ell_1} \le 1, b^*|_X \equiv 0 \right\}$, and hence there exists $b^* \in \overline{B}^*_{(p,q]}$ such that $\|b^* - b_0^*\| \leq \varepsilon_q$. Thus we have that $b^*(x) > b_0^*(x) - \varepsilon_q \geq \frac{1}{4} - \varepsilon_q$. □

Lemma 4.7. *Let $(x_r)_{r=1}^a$ be a skipped block sequence with $a \leq n_{2j}$ and $2j \leq \min ran(x_2)$. Then there exists $\gamma \in \Lambda$ of weight $\frac{1}{m_{2j}}$ such that $\sum_{i=1}^a x_i(\gamma) \geq \frac{1-8\varepsilon}{8m_{2j-1}} \sum_{i=1}^a \|x_i\|_{Z/X}$.*

Proof. Choose $p_0 < p_1 < p_2 < \cdots < p_a$ such that $ran(x_r) \subset (p_{r-1}, p_r)$ for all $1 \leq r \leq a$. By Lemma 4.6 we may choose $b_r^* \in \overline{B}^*_{(p_{r-1},p_r)}$ such that $b_r^*(x_r) \geq \frac{1-8\varepsilon_{p_r-1}}{8}\|x\|_{Z/X}$. By Lemma 4.4 there exists $\xi_r \in \Theta_{p_r}$ for each $1 \leq r \leq a$ with weight $\frac{1}{m_{2j}}$ such that the analysis of $\gamma = \xi_a$ is $(p_r, b_r^*, \xi_r)_{r=1}^a$. We first note that $d_{\xi_i}^*(x_r) = 0$ for all i, r because $\xi_i \in \Theta_{p_i}$ and $p_i \notin ran(x_r)$ for all i, r. We further note that $b_i^*(x_r) = 0$ for all $i \neq r$ because $ran(x_r), ran(b_r^*) \subset (p_{r-1}, p_r)$. We may now obtain a lower estimate for $\sum_{i=1}^a x_i(\gamma)$ by using the evaluation analysis of γ.

$$\begin{aligned}
\sum_{r=1}^a x_r(\gamma) &= e_\gamma^*(\sum_{r=1}^a x_r) \\
&= \sum_{i=1}^a d_i^*(\sum_{r=1}^a x_r) + \frac{1}{m_{2j}} \sum_{i=1}^a b_i^*(\sum_{r=1}^a x_r) \\
&= \frac{1}{m_{2j}} \sum_{r=1}^a b_r^*(x_r) \\
&> \frac{1}{m_{2j}} \sum_{r=1}^a \frac{1 - 8\varepsilon_{p_r-1}}{8} \|x_r\|_{Z/X}
\end{aligned}$$

□

In Definition 2.9 we stated what it meant for a block basis (x_i) of Y to be a $(\mathbf{w}, C)$-RIS for some sequence $\mathbf{w} = (w_i) \subset (0, 1]$. We will now be solely considering the case $w_i = \frac{1}{m_i}$. In the future, we will use the term C-RIS to mean $((m_i^{-1}), C)$-RIS. To simply some proofs we will also add an additional condition to the definition of C-RIS.

Definition 4.8. Let (x_n) be a block basis in Z and $C > 0$. We say (x_n) is a *C-Rapidly Increasing Sequence*, or C-RIS, if for $k \in \mathbb{N}$

(1) $\|x_k\| \leq C$

(2) $|x_k(\gamma)| \leq C weight(\gamma)$ if $k \geq 2$ and $\gamma \in \overline{\Gamma}$ with $weight(\gamma) \geq m^{-1}_{\max \mathrm{rg}(x_{k-1})}$.

(3) $|x_k(\gamma)| \leq C m_1^{-1}$ if $\gamma \in \overline{\Gamma}$ with $weight(\gamma) = m_1^{-1}$

Adding condition (3) is not a significant change as if $(x_k)_{k=1}^\infty$ is a C-RIS for Definition 2.9, then $(x_k)_{k=2}^\infty$ is a C-RIS for Definition 4.8. The new definition will however make the statements and proofs of certain theorems cleaner.

It will be essential for us to obtain certain upper bounds on values of the form $|e_\gamma^* \sum_{k\in I} \lambda_k x_k|$ where (x_k) is a C-RIS. Estimating these bounds for $\gamma \in \overline{\Gamma} \setminus \Gamma$ will follow the same proofs as in [AH].

Lemma 4.9. *Let (x_k) be a C-RIS. If $\gamma \in \overline{\Gamma} \setminus \Gamma$ and $wt(\gamma) = m_i^{-1}$ then,*

$$|e_\gamma^*(x_k)| \le Cm_i^{-1} \qquad \text{if } i < \max ran(x_{k-1}) \text{ or if } i > \max ran(x_k)$$

Proof. The definition of C-RIS trivially gives the estimate that $|e_\gamma^*(x_k)| \le Cm_i^{-1}$ if $i < \max ran(x_{k-1})$. We now consider the case $i > \max ran(x_k)$. The evaluation analysis for γ is given by

$$e_\gamma^* = \sum_{r=1}^{a} d_{\xi_r}^* + \frac{1}{m_i} b_r^*,$$

for some $(\xi_r)_{r=1}^a \subset \overline{\Gamma} \setminus \Gamma$ and $(b_r^*)_{r=1}^a \subset [F_i^*]$. The element ξ_r has weight m_i^{-1}, for each $1 \le r \le a$. This is important as the set Θ_p contains elements of weight m_i^{-1} only if $p \ge i$. Thus if we consider p_1 so that $\xi_1 \in \Theta_{p_1}$ then $p_1 \ge i > \max ran(x_k)$. Furthermore, $\min range(d_r^*), \min range(b_r^*) > p_1 > \max ran(x_k)$ for all $1 < r \le a$. Thus we have that $d_r^*(x_k) = 0$ for all $1 \le r \le a$, and $b_r^*(x_k) = 0$ for all $1 < r \le a$. Applying this to the evaluation analysis for e_γ^* gives the following desired result,

$$|e_\gamma^*(x_k)| = |\sum_{r=1}^{a} d_{\xi_r}^*(x_k) + \frac{1}{m_i} b_r^*(x_k)| = |\frac{1}{m_i} b_1^*(x_k)| \le \frac{1}{m_i} \|b_1^*\| \|x_k\| \le \frac{C}{m_i}$$

□

Lemma 4.10. *Let $(x_k)_{k\in I}$ be a C-RIS, let λ_k be real numbers, and let γ be an element of Γ. There exists a functional $g^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})]$ such that*

(1) *$supp(g^*) \subset I$*
(2) *$|e_\gamma^*(\sum_{k\in I} \lambda_k x_k)| \le Cg^*(\sum_{k\in I} |\lambda_k| e_k)$.*

Proof. We proceed by induction on the rank of $\gamma \in \Gamma$. If $rank(\gamma) = 1$ then

$$e_\gamma^*(\sum_{k\in I} \lambda_k x_k) = \lambda_{k_0} e_\gamma^*(x_{k_0}) \quad \text{ where } k_0 = \min(I).$$

We may thus simply take $g^* = e_{k_0}^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})]$.

We now assume that $\gamma \in \Gamma$ has rank greater than 1 and age a. We assume that the lemma holds for all elements of Γ with rank less than that of γ. We consider the evaluation analysis of e_γ^*, which is given by

$$e_\gamma^* = \sum_{i=1}^{a} d_{\xi_i}^* + m_1^{-1} \sum_{i=1}^{a} m_1 \beta_i e_{\eta_i}^* = \sum_{i=1}^{a} d_{\xi_i}^* + \sum_{i=1}^{a} \beta_i e_{\eta_i}^*,$$

for some sequences $(\xi_i)_{i=1}^a, (\eta_i)_{i=1}^a \subset \Gamma$. Recall that ℓ_0 was chosen so that $age(\eta) \le \ell_0$ for all $\eta \in \Gamma$. Let $(p_i)_{i=0}^a \subset \mathbb{N}$ be the sequence such that $\xi_i \in \Delta_{p_i}$ for all $1 \le i \le a$ and $p_0 = 0$.

Let $I_0 = \{k \in I : p_r \in range(x_k) \text{ for some } 1 \le r \le a\}$. As (x_k) is a block sequence, for each $1 \le r \le a$ there is at most one $k \in I$ such that $p_r \in range(x_k)$. Thus, $|I_0| \le a \le \ell_0$. We then set $I_r = \{k \in I : range(x_k) \subset (p_{r-1}, p_r)\}$. Note that, if $k \notin \cup_{r=0}^{a} I_r$, then $e_\gamma^*(x_k) = 0$. We now have the following equality,

$$e_\gamma^*(\sum \lambda_k x_k) = e_\gamma^*(\sum_{k \in I_0} \lambda_k x_k) + \sum_{r=1}^{a} \beta_r e_{\eta_r}^*(\sum_{k \in I_r} \lambda_k x_k)$$

For $k \in I_0$, we apply condition (3) in the definition of C-RIS to get the estimate $|e_\gamma^*(x_k)| \le Cm_1^{-1}$. Thus we now have that,

$$|e_\gamma^*(\sum \lambda_k x_k)| \le Cm_1^{-1} \sum_{k \in I_0} |\lambda_k| + \sum_{r=1}^{a} |\beta_r e_{\eta_r}^*(\sum_{k \in I_r} \lambda_k x_k)|. \tag{30}$$

For each $1 \le r \le a$, we apply the induction hypothesis to $\eta \in \Gamma$ to obtain $g_r^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})]$ with $supp(g_r^*) \subset I_r$, such that

$$|e_{\eta_r}^*(\sum_{k \in I_r} \lambda_k x_k)| \le Cg_r^*(\sum_{k \in I_r} |\lambda_k| e_k)). \tag{31}$$

We now define g^* by setting

$$g^* = m_1^{-1}(\sum_{k \in I_0} e_k^* + \sum_{r=1}^{a} g_r^*)$$

This is a sum, weighted by m_1^{-1} of at most $2l_0 \le n_0$ functionals in $W[(\mathcal{A}_{n_1}, m_1^{-1})]$ which are each supported on disjoint intervals. Thus $g^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})]$. We now use (30) and (31) to obtain the following.

$$\begin{aligned}
|e_\gamma^*(\sum \lambda_k x_k)| &\le Cm_1^{-1} \sum_{k \in I_0} |\lambda_k| + \sum_{r=1}^{a} |\beta_r e_{\eta_r}^*(\sum_{k \in I_r} \lambda_k x_k)| \quad \text{by (30)} \\
&\le Cm_1^{-1} \sum_{k \in I_0} |\lambda_k| + \sum_{r=1}^{a} |\beta_r| C g_r^*(\sum_{k \in I_r} |\lambda_k| e_k) \quad \text{by (31)} \\
&\le Cm_1^{-1} \sum_{k \in I_0} |\lambda_k| + \sum_{r=1}^{a} m_1^{-1} C g_r^*(\sum_{k \in I_r} |\lambda_k| e_k) \quad \text{as } |\beta_r| \le m_1^{-1} \\
&= Cm_1^{-1}(\sum_{k \in I_0} e_k^* + \sum_{r=1}^{a} g_r^*)(\sum_{k \in I_0} |\lambda_k| e_k + \sum_{r=1}^{a} \sum_{k \in I_r} |\lambda_k| e_k) \quad \text{as } g_r^* = 0 \text{ or } supp(g_r^*) \subset I_r \\
&= Cg^*(\sum_{k \in I} |\lambda_k| e_k)
\end{aligned}$$

□

Proposition 4.11 (Basic Inequality)**.** *Let* $(x_k)_{k\in I}$ *be a* C*-RIS, let* λ_k *be real numbers, and let* γ *be an element of* $\overline{\Gamma}$*. There exists* $k_0 \in I$ *and a functional* $g^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ *such that:*

(1) *either* $g^* = 0$ *or* $weight(g^*) = weight(\gamma)$ *and* $supp(g^*) \subset \{k \in I : k > k_0\}$

(2) $|e^*_\gamma(\sum_{k\in I} \lambda_k x_k)| \leq 2C|\lambda_{k_0}| + Cg^*(\sum_{k\in I} |\lambda_k| e_k)$

Moreover, if j_0 *is such that* $|e^*_\xi(\sum_{k\in I} \lambda_k x_k)| \leq C \max_{k\in J} |\lambda_k|$ *for all subintervals* J *of* I *and all* $\xi \in \overline{\Gamma}$ *of weight* $m_{j_0}^{-1}$*, then we may choose* g^* *to be in* $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$*.*

Proof. We assume that $\gamma \in \Gamma$ and will show that the moreover part holds. For $\gamma \in \Gamma$, the rest of the proposition is an obvious corollary of Lemma 4.10. We first consider $j_0 \neq 1$. By Lemma 4.10, there exists $g^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})]$ satisfying (1) and (2). Thus $g^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})] \subset W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$, which proves the moreover part. We now consider $j_0 = 1$, and assume that $|e^*_\xi(\sum_{k\in I} \lambda_k x_k)| \leq C \max_{k\in J} |\lambda_k|$ for all subintervals J of I and all $\xi \in \overline{\Gamma}$ of weight m_1^{-1}. However, γ has weight m_1^{-1}, and thus we may take $g^* = 0 \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$. Thus the proposition is true for all $\gamma \in \Gamma$.

We now consider the case $\gamma \in \overline{\Gamma} \setminus \Gamma$, and will proceed by induction on the rank of γ. There are no $\gamma \in \overline{\Gamma} \setminus \Gamma$, and thus we first consider the case that $rank(\gamma) = 2$. We then have that

$$e^*_\gamma(\sum_{k\in I} \lambda_k x_k) = \lambda_{k_0} e^*_\gamma(x_{k_0}) + \lambda_k e^*_\gamma(x_{k_1}),$$

where k_0 and k_1 are the first two elements of I. Thus setting $g^* = e^*_{k_1}$ gives the desired inequality.

We now assume that $\gamma \in \overline{\Gamma} \setminus \Gamma$ has rank greater than 2, age a, and weight m_h^{-1}. We suppose that there is some $\ell \in I$ such that $\max range(x_\ell) < h \leq \max range(x_{\ell+1})$. The simpler cases of $h \leq \max range(x_1)$ or $\max rang(x_k) < h$ for all $k \in I$ can be proved in the same way, and so will not be considered. We will split the following summation into three parts, and estimate each part separately.

$$e^*_\gamma(\sum_{k\in I} \lambda_k x_k) = \sum_{k\in I, k<\ell} \lambda_k e^*_\gamma(x_k) + \lambda_\ell e^*_\gamma(x_\ell) + e^*_\gamma(\sum_{k\in I, k>\ell} \lambda_k x_k). \tag{32}$$

For the first part, we have by our choice of ℓ that $h > \max range(x_k)$ for all $k < \ell$. Thus Lemma 4.9 gives us $|e^*_\gamma(x_k)| \leq Cm_h^{-1}$. Furthermore, the inequality $\max range(x_\ell) < h$ implies that $\sharp\{k \in I : k < \ell\} < h$, and thus trivially $\sharp\{k \in I : k < \ell\} m_h^{-1} < 1$. We now have the following upper bound.

$$|\sum_{k\in I, k<\ell} \lambda_k e^*_\gamma(x_k)| \leq C \sum_{k<\ell} m_h^{-1}||\lambda_k| \leq C\sharp\{k \in I : k < \ell\} m_h^{-1} \max_{k<\ell} |\lambda_k| < C \max_{k<\ell} |\lambda_k|$$

For the second term we have the trivial bound

$$|\lambda_\ell e^*_\gamma(x_\ell)| \leq C|\lambda_\ell|.$$

Thus combining the first two terms gives the inequality

$$|e_\gamma^*(\sum_{k\in I,k\le \ell}\lambda_k x_k)|\le C\max_{k\in I,k<\ell}|\lambda_k|+C|\lambda_\ell|\le 2C|\lambda_{k_0}|, \tag{33}$$

for some particular $k_0\le \ell$.

We now set $I'=\{k\in I: k>\ell\}$, and focus on estimating the last term: $|e_\gamma^*(\sum_{k\in I'}\lambda_k x_k)|$. The evaluation analysis of e_γ^* is given by

$$e_\gamma^*=\sum_{r=1}^a d_{\xi_r}^*+m_h^{-1}\sum_{r=1}^a b_r^*,$$

for some $(\xi_r)_{r=1}^a\subset\overline{\Gamma}\setminus\Gamma$ and $(b_r^*)_{r=1}^a\subset[F_i^*]$. We denote $(p_r)_{r=1}^a\subset\mathbb{N}$ to be the sequence such that $\xi_r\in\Theta_{p_r}$ for each $1\le r\le a$. This implies that $b_r^*\in\overline{B}^*_{(p_{r-1},p_r)}$ for all $1\le r\le a$, where we set $p_0=0$. Let $I_0'=\{k\in I: p_r\in range(x_k)$ for some $1\le r\le a\}$. As (x_k) is a block sequence, for each $1\le r\le a$ there is at most one $k\in I'$ such that $p_r\in range(x_k)$. We then set $I_r'=\{k\in I: range(x_k)\subset(p_{r-1},p_r)\}$. Note that, if $k\notin\cup_{r=0}^a I_r$, then $e_\gamma^*(x_k)=0$. We now have the following equality,

$$e_\gamma^*(\sum\lambda_k x_k)=\sum_{k\in I_0}\lambda_k e_\gamma^*(x_k)+m_h^{-1}\sum_{r=1}^a b_r^*(\sum_{k\in I_r}\lambda_k x_k) \tag{34}$$

As $b_r^*\in S_{\ell_1(\overline{\Gamma})}\cap\oplus_{p_{r-1}+1}^{p_r-1}F_i^*$, the functional b_r^* is a convex combination of functionals $\pm e_\eta^*$ with $p_{r-1}<rank(\eta)<p_r$. Thus we may choose η_r to be such an η with

$$|b_r^*(\sum_{k\in I_r'}\lambda_k x_k)|\le|e_{\eta_r}^*(\sum_{k\in I_r'}\lambda_k x_k)|.$$

For each r, we apply the induction hypothesis to each element $\eta_r\in\overline{\Gamma}$ and the C-RIS $(x_k)_{k\in I_r'}$, obtaining $k_r\in I_r'$ and a functional $g_r^*\in W[(\mathcal{A}_{3n_j},m_j^{-1})_{j\in\mathbb{N}}]$ supported on $\{k\in I_r': k>k_r\}$ satisfying

$$|e_{\eta_r}^*(\sum_{k\in I_r'}\lambda_k x_k)|\le 2C|\lambda_{k_r}|+Cg_r^*(\sum_{k\in I_r'}|\lambda_k|e_k) \tag{35}$$

We now define g^* by setting

$$g^*=m_h^{-1}(\sum_{k\in I_0'}e_k^*+\sum_{r=1}^a(e_{k_r}^*+g_r^*)) \tag{36}$$

This is a sum, weighted by m_h^{-1} of at most $3n_h$ functionals in $W[(\mathcal{A}_{n_1},m_1^{-1})]$ which are each supported on disjoint intervals. Thus $g^*\in W[(\mathcal{A}_{n_1},m_1^{-1})]$. We now use (32), (33), (34) and

(35) to obtain the following.

$$\begin{aligned}|e^*_\gamma(\sum \lambda_k x_k)| &\le 2C|\lambda_{k_0}| + Cm_h^{-1}\sum_{k\in I_0'}|\lambda_k| + m_h^{-1}\sum_{r=1}^{a} b^*_r(\sum_{k\in I_r'}\lambda_k x_k)| \quad \text{by (33)}\\ &\le 2C|\lambda_{k_0}| + Cm_h^{-1}\sum_{k\in I_0}|\lambda_k| + m_h^{-1}\sum_{r=1}^{a} |e^*_{\eta_r}(\sum_{k\in I_r'}\lambda_k x_k)| \quad \text{by (34)}\\ &\le 2C|\lambda_{k_0}| + Cm_h^{-1}(\sum_{k\in I_0}|\lambda_k| + \sum_{r=1}^{a}|\lambda_{k_r}| + g^*_r(\sum_{k\in I_r'}\|\lambda_k|e_k)) \quad \text{by (35)}\\ &=\le 2C|\lambda_{k_0}| + Cg^*(\sum_{k\in I'}|\lambda_k|e_k)\end{aligned}$$

If j_0 satisfies the moreover condition in the statement of the proposition we proceed by the same induction. The base case is the same. When we prove the inductive step for γ with weight m_h^{-1} we need to consider separately the cases $h \ne j_0$ and $h = j_0$. For the case $h \ne j_0$, the proof remains unchanged as we are able to assume by induction that $g^*_r \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\ne j_0}]$. Thus when we define g^* as in (36) we have that $g^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\ne j_0}]$.

For the remaining case $h = j_0$, the moreover assumption gives automatically that $|e^*_\gamma(\sum_{k\in I}\lambda_k x_k)| \le C\max_{k\in I}|\lambda_k|$. Thus we are able to take $g^* = 0$. □

The basic inequality relates the functionals e^*_γ with $\gamma \in \overline{\Gamma}$ to functionals in the mixed Tsirelson space $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$. The following proposition gives us good estimates for what happens when we apply functionals in mixed Tsirelson spaces to averages.

Proposition 4.12. *If $j_0 \in \mathbb{N}$ and $f^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ is an element of weight m_h^{-1}, then*

$$|f^*(n_{j_0}^{-1}\sum_{\ell=1}^{n_{j_0}} e_\ell)| \le \begin{cases} 2m_h^{-1}m_{j_0}^{-1} & \text{if } h < j_0\\ m_h^{-1} & \text{if } h \ge j_0\end{cases}$$

In particular, the norm of $n_{j_0}^{-1}\sum_{\ell=1}^{n_{j_0}} e_\ell$ in $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ is exactly $m_{j_0}^{-1}$. If we make the additional assumption that $f^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\ne j_0}$ then

$$|f^*(n_{j_0}^{-1}\sum_{\ell=1}^{n_{j_0}} e_\ell)| \le \begin{cases} 2m_h^{-1}m_{j_0}^{-2} & \text{if } h < j_0\\ m_h^{-1} & \text{if } h > j_0\end{cases}$$

In particular, the norm of $n_{j_0}^{-1}\sum_{\ell=1}^{n_{j_0}} e_\ell$ in $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\ne j_0}]$ is at most $m_{j_0}^{-2}$.

We now are able to combine the basic inequality with Proposition 4.12 to obtain the following lemma which will be used extensively in proving our main result.

Lemma 4.13. *Let $\bar{x} = (x_k)_{k=1}^{n_{j_0}}$ be a skipped block C-RIS with $j_0 > 1$. Then $z(j_0, \bar{x}) = z = \frac{m_{j_0}}{n_{j_0}}\sum_{k=1}^{n_{j_0}} x_k$ has the following four properties.*

(1) $d_\xi^*(z) \le \frac{3Cm_{j_0}}{n_{j_0}} < \frac{C}{m_{j_0}}$ *for all* $\xi \in \overline{\Gamma}$
(2) $\|z\| < 2C$
(3) *For all* $\gamma \in \overline{\Gamma} \setminus \Gamma$ *with weight* $\frac{1}{m_h}$ *such that* $h \neq j_0$ *we have*

$$|z(\gamma)| \le \begin{cases} 3Cm_h^{-1} & \text{if } h < j_0, \\ 2Cm_{j_0}^{-1}, & \text{if } h > j_0. \end{cases}$$

(4) *For all* $\gamma \in \Gamma$ *we have that* $|z(\gamma)| < Cm_{j_0}^{-1}$

Proof. Let $\xi \in \overline{\Gamma}$. We have that (x_k) is a block sequence and hence $d_\xi^*(\sum x_k) = d_\xi^*(x_m)$ for some $1 \le m \le n_{j_0}$. The inequality in (1) is given by $|d_\xi^*(\frac{m_{j_0}}{n_{j_0}} \sum_{k=1}^{n_{j_0}} x_k)| = |d_\xi^* \frac{m_{j_0}}{n_{j_0}} x_m)| \le \|d_\xi^*\| \frac{m_{j_0}}{n_{j_0}} \|x_k\| \le \frac{3Cm_{j_0}}{n_{j_0}}$.

To obtain the inequality in (2) we choose $\gamma \in \overline{\Gamma}$ then apply the basic inequality to $e_\gamma^*(z)$ to obtain

$$\begin{aligned} |e_\gamma^*(z)| &\le 2m_{j_0} n_{j_0}^{-1} C + Cm_{j_0} g^*(n_{j_0}^{-1} \sum_{i=1}^{n_{j_0}} e_k) \\ &\le 2m_{j_0} n_{j_0}^{-1} C + C < 2C \qquad \text{by Proposition 4.12} \end{aligned}$$

Thus we have that $\|z\| = \sup_{\gamma \in \overline{\Gamma}} |e_\gamma^*(z)| < 2C$.

To obtain the inequality in (3) we apply the basic inequality to $\gamma \in \overline{\Gamma}$ with weight m_h^{-1} such that $h \neq j_0$. This gives $g^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j \in \mathbb{N}}]$ with $g^* = 0$ or $weight(g^*) = m_h^{-1}$ such that

$$|e_\gamma^*(z)| \le 2m_{j_0} n_{j_0}^{-1} C + Cm_{j_0} g^*(n_{j_0}^{-1} \sum_{i=1}^{n_{j_0}} e_k)$$

We now apply Proposition 4.12 to g^* to obtain

$$g^*(n_{j_0}^{-1} \sum_{i=1}^{n_{j_0}} e_k) \le \begin{cases} 2m_h^{-1} m_{j_0}^{-1} & \text{if } h < j_0; \\ m_h^{-1} & \text{if } h > j_0. \end{cases}$$

Combining the above two inequalities gives (3) as $2m_{j_0} n_{j_0}^{-1} \le m_{j_0}^{-1}$ by (26).

For our final inequality (4) we apply Lemma 4.10 to $\gamma \in \Gamma$ to obtain $g^* \in W[(\mathcal{A}_{n_1}, m_1^{-1})]$ such that

$$|e_\gamma^*(z)| \le Cm_{j_0} g^*(n_{j_0}^{-1} \sum_{i=1}^{n_{j_0}} e_k)$$

We now apply Proposition 4.12 to g^* to obtain

$$g^*(n_{j_0}^{-1} \sum_{i=1}^{n_{j_0}} e_k) \le 2m_1^{-1} m_{j_0}^{-1} \qquad \text{as } 1 < j_0$$

Combining the above two inequalities gives (4) as $2m_1^{-1} < 1$. □

Proposition 4.14. *The FDD $(\bar{F}_i)$ of Z is shrinking and hence Z^* is isomorphic to ℓ_1.*

Proof. The Banach space Z is a separable $\mathcal{L}_\infty$ Banach space, and thus the dual of Z is isomorphic to ℓ_1 if and only if ℓ_1 does not embed into Z [LS]. Thus if $(\bar{F}_i)$ is shrinking then Z^* is isomorphic to ℓ_1. We assume to the contrary that there exists a normalized block basis (b_k) which is not weakly null. Hence there exists $f \in Z^*$ such that $|f(b_k)| \not\to 0$. By Proposition 2.8 there also exists some skipped block C-RIS (x_k) such that $|f(x_k)| \not\to 0$. After passing to a subsequence we may assume that $|f(x_k)| > \delta$ for all $k \in \mathbb{N}$ and some $\delta > 0$. In particular we have that $|f(n_{2j}^{-1} \sum_{k=1}^{n_{2j}} x_k)| > \delta$ for all $j \in \mathbb{N}$. However, by Lemma 4.13 (2) we have that $\|n_{2j}^{-1} \sum_{k=1}^{n_{2j}} x_k\| \leq 2Cm_{2j}^{-1}$. This is a contradiction if $j \in \mathbb{N}$ is chosen to be sufficiently large. □

We are now prepared to prove our main result.

Proof of 3.5. By Theorem 2.12 we just need to prove that if (x_n) is a C-RIS then $\lim_{n\to\infty} dist(T(x_n), [x_n]+X) = 0$. We assume to the contrary that there is some $C > 1$ and a C-RIS (x_n) with $\|T(x_n)\|_{Z/X+[x_n]} \geq 8 + 8\varepsilon$. As (x_n) is a block sequence of a shrinking FDD, we may pass to a subsequence of (x_n) and a compact perturbation of T so that there exists integers $0 = p_0 < p_1 < p_1 + 1 < p_2 < p_2 + 1 < p_3 \cdots$ with $ran(x_n), ran(T(x_n)) \subset (p_{n-1}+1, p_n)$ for all $n \in \mathbb{N}$. Following the proof of Lemma 4.6 we may choose for each $n \in \mathbb{N}$ a function $b_n^* \in \overline{B}^*_{(p_{n-1}+1, p_n-1]}$ such that $|b_n^*(x_n)| < \varepsilon_n$ and $b_n^*(T(x_n)) \geq 1$. We recall from Lemma 4.13 that we denote

$$z(j, (x_i)) = \frac{m_j}{n_j} \sum_{k=1}^{n_j} x_k.$$

We now fix some $i_0 \in \mathbb{N}$. The proof will proceed by constructing a block sequence $(u_i)_{i=1}^{n_{2i_0-1}}$ of (x_i) with each u_i being of the form $z(j, \bar{x}^i)$ for some $j \in \mathbb{N}$ and subsequence $\bar{x}^i$ is a subsequence of (x_n). First we choose j_1 such that $m_{4j_1-2} > n_{2i_0-1}^2$ and $k_1 \in \mathbb{N}$ so that $4j_1 - 2 \leq p_{k_1}$. We then set $u_1 = z(4j_1 - 2, (x_i)_{i\geq k_1})$. By Lemma 4.4 we may choose $\gamma_1 \in \overline{\Gamma}$ with $rank(\gamma_1) > \max supp(u_1, T(u_1))$ such that the analysis of γ_1 is $(p_r, b_r^*, \xi_r)_{k_1\leq r\leq k_1+n_{4j_1-2}-1}$ for some $\xi_r \in \Theta_{p_r}$. A simple calculation shows that $e^*_{\gamma_1}(T(u_1)) \geq 1$ and $|e^*_{\gamma_1}(u_1)| < \varepsilon_1$. We now inductively construct u_r and γ_r for $2 \leq r \leq n_{2i_0-1}$. We first set $j_r = \sigma(\gamma_{r-1})$ and choose $k_r \in \mathbb{N}$ such that $4j_r < p_{k_r}$. We then set $u_r = z(4j_r, (x_i)_{i=k_r}^\infty)$. Again by Lemma 4.4 we may choose $\gamma_r \in \overline{\Gamma}$ of weight $\frac{1}{m_{4j_r}}$ with $rank(\gamma_r) > \max supp(u_r, T(u_r))$ such that $e^*_{\gamma_r}(T(u_r)) \geq 1$ and $|e^*_{\gamma_r}(u_r)| < \varepsilon_r$. This completes the construction of $(u_r)_{r=1}^{n_{2i_0-1}}$. We now set $u = z(2i_0 - 1, (u_r))$. Note that we have chosen $(\gamma_r)_{r=1}^{n_{2i_0-1}}$ and $(j_r)_{r=1}^{n_{2i_0-1}}$ to satisfy the conditions of Lemma 4.5, and thus there exists $\gamma \in \overline{\Gamma}$ with analysis $(p_{k_r}+1, e^*_{\gamma_r}, e^*_{\xi_r})$ for some $\xi_r \in \Theta_{p_{k_r}+1}$ with weight $\frac{1}{m_{2j_0-1}}$. A simple calculation shows that $e^*_\gamma(T(u)) \geq 1$ and $e^*_\gamma(u) < \varepsilon$. We will prove that actually $\|u\| \leq 9Cm_{2j_0-1}^{-1}$. Thus by choosing i_0 to be sufficiently large we reach a contradiction with $\|T\|$ being bounded.

The norm of u is given by $\|u\| = \max_{\gamma\in\overline{\Gamma}} |u(\gamma)|$. By part (5) of Lemma 4.13, we have that $|u(\gamma)| \leq Cm_{2i_0-1}^{-1}$ for all $\gamma \in \Gamma$. We will prove that $|u(\gamma)| \leq 9Cm_{2j_0-1}^{-1}$ for all $\gamma \in \overline{\Gamma}\setminus\Gamma$ through the moreover part of Proposition 4.11. We first note that parts (2) and (3) of Lemma 4.13 imply that the sequence (u_r) is a $3C$-RIS. Assuming we are able to satisfy the moreover part of Proposition 4.11, we would have that

$$\begin{aligned}\|u\| =& \left\| \frac{m_{2i_0-1}}{n_{2i_0-1}} \sum_{r=1}^{n_{2i_0-1}} u_r \right\| \\ \leq& 6C\frac{m_{2i_0-1}}{n_{2i_0-1}} + 3Cg^*\left(\frac{m_{2i_0-1}}{n_{2i_0-1}} \sum_{r=1}^{n_{2i_0-1}} e_r\right) \quad \text{for some } g^* \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq 2i_0-1}] \\ \leq& 6C\frac{m_{2i_0-1}}{n_{2i_0-1}} + \frac{3C}{m_{2i_0-1}} \\ \leq& \frac{6C}{m_{2i_0-1}} + \frac{3C}{m_{2i_0-1}}.\end{aligned}$$

Thus all that remains to be verified is the moreover part of Proposition 4.11. Given a subinterval J of $[1, n_{2i_0-1}]$ and an element $\gamma' \in \overline{\Gamma}\setminus\Gamma$ of weight $m_{2i_0-1}^{-1}$ we need to prove that $|e^*_{\gamma'}(\sum_{r\in J} u_r)| \leq 3C$. Without loss of generality we may assume that the age of γ' is the maximal value n_{2i_0-1}. We denote the analysis of γ' by $(q'_r, e^*_{\gamma'_r}, e^*_{\xi'_r})_{r\leq n_{2i_0-1}}$ and the analysis of γ by $(q_r, e^*_{\gamma_r}, e^*_{\xi_r})_{r\leq n_{2i_0-1}}$. We thus have the following evaluation analysis for γ',

$$e^*_{\gamma'} = \sum_{r=1}^{n_{2i_0-1}} d^*_{\xi'_r} + \frac{1}{m_{2i_0-1}} e^*_{\gamma'_r}.$$

By the definition of Θ_n, it must be that $wt(\gamma'_r), wt(\gamma_r) < n_{2i_0-1}^{-2}$ for all $1 \leq r \leq n_{2i_0-1}$. This important fact will be used repeatedly in the remainder of the proof. Because (u_i) is a block sequence, there exists $1 \leq j \leq n_{2i_0-1}$ such that $d^*_{\xi_r}(\sum_{i\in J} u_i) = d^*_{\xi_r}(u_j)$. By applying this fact with part (1) of Lemma 4.13 we obtain the following inequality.

$$|d^*_{\xi_r}(\sum_{i\in J} u_i)| = |d^*_{\xi_r}(u_j)| \leq Cwt(\gamma_j) < Cn_{2i_0-1}^{-2} \quad \text{for all } 1 \leq r \leq n_{2i_0-1}. \tag{37}$$

By Proposition 4.3 there exists $1 \leq \ell \leq n_{2i_0-1}$ such that $\xi'_r = \xi_r$ for all $r < \ell$ and $wt(\gamma'_j) \neq wt(\gamma_r)$ for all j and all $\ell < r \leq n_{2i_0-1}$. In particular $\gamma'_r = \gamma_r$ and $q'_r = q_r$ for all $r < \ell$. Thus we have that

$$|e^*_{\gamma'_r}(\sum_{i\in J} u_i)| = |e^*_{\gamma_r}(\sum_{i\in J} u_i)| = |e^*_{\gamma_r}(u_r)| < \varepsilon_{p_{k_{r-1}}} \quad \text{for all } r < \ell. \tag{38}$$

Part (2) of Lemma 4.13 implies the following.

$$|e^*_{\gamma'_\ell}(u_j)| \leq \|u_j\| \leq 2C \quad \text{if } wt(\gamma'_\ell) = wt(\gamma_j). \tag{39}$$

We use part (3) of Lemma 4.13 with the fact that $wt(\gamma'_r), wt(\gamma_r) < n_{2i_0-1}^{-2}$ for all $1 \le r \le n_{2i_0-1}$ to achieve

$$|e^*_{\gamma'_r}(u_j)| \le 3Cn_{2i_0-1}^{-2} \quad \text{if } wt(\gamma'_r) \ne wt(\gamma_j). \tag{40}$$

We will apply Inequality (40) for all $r > \ell$ and for the case $r = \ell$ with $wt(\gamma'_\ell) \ne wt(\gamma_j)$. The sequence $(e^*_{\gamma'_r})_{1\le r\le n_{2i_0-1}}$ is a block sequence of $(\bar{F}^*_i)$ and $(u_i)_{1\le i\le n_{2i_0-1}}$ is a block sequence of $(\bar{F}_i)$. This implies the following simple combinatorial result.

$$\sharp\{(r,j)|e^*_{\gamma'_r}(u_j) \ne 0\} < 2n_{2i_0-1}. \tag{41}$$

Combining all the inequalities (37),(38),(39),(40), and (41) gives our desired estimate.

$$\begin{aligned}
|e^*_{\gamma'}(\sum_{i\in J} u_i)| &= \sum_{r=1}^{n_{2i_0-1}} d^*_{\xi'_r}(\sum_{i\in J} u_i) + m_{2i_0-1}^{-1} \sum_{r=1}^{n_{2i_0-1}} e^*_{\gamma'_r}(\sum_{i\in J} u_i)\\
&= \left(\sum_{r=1}^{n_{2i_0-1}} d^*_{\xi'_r} + m_{2i_0-1}^{-1}\left(\sum_{r<\ell} e^*_{\gamma'_r} + e^*_{\gamma'_\ell} + \sum_{r>\ell} m_{2i_0-1}^{-1} e^*_{\gamma'_r}\right)\right)(\sum_{i\in J} u_i)\\
&< Cn_{2i_0-1}^{-1} + m_{2i_0-1}^{-1}\varepsilon + m_{2i_0-1}^{-1}2C + m_{2i_0-1}^{-1}2n_{2i_0-1}3Cn_{2i_0-1}^{-2}\\
&< C.
\end{aligned}$$

Thus the moreover part of Proposition 4.11 has been verified, and the proof is complete. ☐

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