

# THE LEBESGUE MEASURE AND THE LEBESGUE INTEGRAL

(An elementary introduction )

We are developing an elementary approach to the theory of the Lebesgue measure and the Lebesgue integral on the real line  $\mathbb{R}$ .

When you do an exercise you may (and often should) use facts shown prior to that exercise, but you are not allowed to use future facts.

## 1. INTRODUCTION AND MOTIVATION

Assume  $f : [a, b] \rightarrow \mathbb{R}$  is a strictly increasing and, to make things easier, continuous function.

In order to compute  $\int_a^b f(x)dx$  we could partition the image (which is the interval  $[f(a), f(b)]$ ) of  $f$  into subintervals instead of subdividing the domain  $[a, b]$ .

Let  $Q = (y_0, y_1, \dots, y_n)$ ,  $f(a) = y_0 < y_1 < \dots < y_n = f(b)$  a partition of  $[f(a), f(b)]$ . The pre-image of the interval  $[y_{i-1}, y_i]$  is easy to compute since  $f$  is strictly increasing:

$$f^{-1}([y_{i-1}, y_i]) = \{x \in [a, b] : y_{i-1} \leq f(x) \leq y_i\} = [f^{-1}(y_{i-1}), f^{-1}(y_i)].$$

We now can define the upper and the lower sum with respect to the partition  $a = f^{-1}(y_0) < f^{-1}(y_1) < \dots < f^{-1}(y_n) = b$ .

Define  $P = (x_0, x_1, \dots, x_n) = (f^{-1}(y_0), f^{-1}(y_1), \dots, f^{-1}(y_n))$  (pre-image of  $Q$ ).

$$S^-(P, f) = \sum_{i=1}^n f(x_{i-1})(x_i - x_{i-1}) = \sum_{i=1}^n y_{i-1}(f^{-1}(y_i) - f^{-1}(y_{i-1})).$$

$$S^+(P, f) = \sum_{i=1}^n f(x_i)(x_i - x_{i-1}) = \sum_{i=1}^n y_i(f^{-1}(y_i) - f^{-1}(y_{i-1})).$$

Since  $f^{-1}(y_i) - f^{-1}(y_{i-1})$  is the length of the pre-image of the interval of  $[y_{i-1}, y_i]$  we can write  $S^-(P, f)$  and  $S^+(P, f)$  as

$$S^-(P, f) = \sum_{i=1}^n y_{i-1} \text{length}(f^{-1}[y_{i-1}, y_i])$$

$$S^+(P, f) = \sum_{i=1}^n y_i \text{length}(f^{-1}[y_{i-1}, y_i])$$

**Remark.** Let us observe an interesting fact. By subdividing the image instead of the domain we get a better, and more direct, error estimate.

Indeed, let us assume we want our error, namely the number  $S^+(P, f) - S^-(P, f)$  to be at most  $\varepsilon > 0$ . We simply choose the mesh of the partition  $Q$  so that

$$\text{mesh} = \max_{i=1,2,\dots,n} y_i - y_{i-1} < \frac{\varepsilon}{b-a}.$$

Then we conclude that

$$\begin{aligned} S^+(P, f) - S^-(P, f) &= \sum_{i=1}^n (y_i - y_{i-1})(f^{-1}(y_i) - f^{-1}(y_{i-1})) \\ &\leq \frac{\varepsilon}{b-a} \sum_{i=1}^n f^{-1}(y_i) - f^{-1}(y_{i-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon. \end{aligned}$$

In particular, we showed that if we let the mesh of  $Q$  converge to zero, then the corresponding upper sums  $S^+(P, f)$  and the lower sums  $S^-(P, f)$  ( $P$  defined as above) converge to  $\int_a^b f(x)dx$ .

Unfortunately this approach (of starting by subdividing the image instead of the domain) does not work that easily for more general functions, i.e. functions which are not monotone.

Let  $f : [a, b] \rightarrow \mathbb{R}$  be any bounded function and let  $A = \inf_{x \in [a, b]} f(x)$  and  $B = \sup_{x \in [a, b]} f(x)$ . We still could consider a partition of  $[A, B]$  say  $Q = (y_0, y_1, \dots, y_n)$ . But now the pre-image  $f^{-1}[y_{i-1}, y_i]$  might not be an interval anymore.

For “nice” function, i.e. functions which change their direction only finitely many times, the pre-image is a finite union of disjoint intervals. But only think of the function  $f(x) = \sin(1/x)$ , for  $x > 0$ , and ask yourself what the pre-image of  $[-1/2, 1/2]$  is.

So we have to solve the problem: find the *length* (we will say *measure*) of subsets which are more general than intervals and even more general than finite unions of intervals.

Then we could define as before

$$S^-(P, f) = \sum_{i=1}^n y_{i-1} \text{length}(f^{-1}[y_{i-1}, y_i]),$$

$$S^+(P, f) = \sum_{i=1}^n y_i \text{length}(f^{-1}[y_{i-1}, y_i])$$

and define

$$\int_a^b f(x) dx = \sup_{Q=(y_0, y_1, \dots, y_n) \in \mathcal{P}_{[A, B]}} S^-(P, f) = \inf_{Q=(y_0, y_1, \dots, y_n) \in \mathcal{P}_{[A, B]}} S^+(P, f).$$

## 2. EXTENSION OF THE NOTION *length* TO MORE GENERAL SETS

First the measure of any interval, open, closed, or half open, is its length.

$$(1) \quad m([a, b]) = m((a, b]) = m([a, b)) = m((a, b)) = b - a \text{ for } a \leq b.$$

Secondly we will consider finite unions of intervals. For the moment we only consider finite unions of open intervals

We need a lemma.

**Lemma 1.** *Let  $U \subset \mathbb{R}$  be the union of (not necessarily disjoint) open intervals  $I_1, I_2, \dots, I_n$ .*

*$I_\ell = (a_\ell, b_\ell)$ , for  $\ell = 1, \dots, n$ .*

*Then there are pairwise disjoint open and non empty intervals  $J_1, J_2, \dots, J_m$ ,  $m \leq n$ , so that  $U = \bigcup_{\ell=1}^m J_\ell$ .*

*Moreover, the  $J_1, J_2, \dots, J_m$  are unique in the following sense:*

*If  $U = \bigcup_{\ell=1}^{\tilde{m}} \tilde{J}_\ell$ , where  $\tilde{J}_1, \tilde{J}_2, \dots, \tilde{J}_{\tilde{m}}$  are disjoint open and non empty.*

*Then it follows that  $m = \tilde{m}$  and that there is a bijection  $b : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, \tilde{m}\}$  so that  $J_i = \tilde{J}_{b(i)}$ .*

*Proof.* By induction on  $n$ . □

**Exercise 1.** *Proof Lemma 1.*

Using Lemma 1, we can define the measure of a finite union of open bounded intervals.

(2) Let  $U$  be the finite union of open, bounded intervals and let  $J_1, J_2, \dots, J_m$  be the (unique) pairwise disjoint, non-empty, open

intervals so that  $U = \bigcup_{\ell=1}^m J_\ell$ .

Then we put

$$m(U) = \sum_{\ell=1}^m m(J_\ell)$$

So we defined up to now *the measure* as a map

$$\mathcal{F} = \{U : U \text{ is finite union of open, bounded intervals}\} \rightarrow [0, \infty).$$

Before we go on to more general sets, we want to observe some easy facts:

**Proposition 2.** *The map  $m : \mathcal{F} \rightarrow [0, \infty)$  has the following properties.*

- a)  $m$  is monotone: If  $U, V \in \mathcal{F}$  and  $U \subset V$ , then  $m(U) \leq m(V)$ .
- b)  $m$  is additive on disjoint sets: If  $U, V \in \mathcal{F}$ , and  $U \cap V = \emptyset$  then  $m(U \cup V) = m(U) + m(V)$ .
- c) If  $U, V \in \mathcal{F}$  then  $m(U \cup V) = m(U) + m(V) - m(U \cap V)$ .  
In particular  $m$  is subadditive:  $m(U \cup V) \leq m(U) + m(V)$  for all  $U, V \in \mathcal{F}$ .

*Proof.* The first two claims, (a) and (b), are easy and are left as an exercise (see below). In order to show (c) we first show the claim under the assumption that  $V$  is an open interval. Thus let

$$V = (A, B) \text{ and } U = \bigcup_{i=1}^n (a_i, b_i), \text{ with } a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n.$$

If  $(A, B) \cap U = \emptyset$  then our claim follows already from part (b). So we can assume that  $(A, B) \cap U \neq \emptyset$ . We define

$$m_1 = \min\{i \leq n : (a_i, b_i) \cap (A, B) \neq \emptyset\} \text{ and } m_2 = \max\{i \leq n : (a_i, b_i) \cap (A, B) \neq \emptyset\}.$$

We distinguish between the following cases:

Case 1:  $a_{m_1} \leq A < B \leq b_{m_2}$  and  $m_1 < m_2$ . Then (draw a picture!)

$$\begin{aligned} m(U \cup V) &= \sum_{i=1}^{m_1-1} (b_i - a_i) + (b_{m_2} - a_{m_1}) + \sum_{i=m_2+1}^n (b_i - a_i) \\ &= \sum_{i=1}^n (b_i - a_i) - \sum_{i=m_1}^{m_2} (b_i - a_i) + (b_{m_2} - a_{m_1}) \\ &= \sum_{i=1}^n (b_i - a_i) + B - A + b_{m_2} - B + A - a_{m_1} - \sum_{i=m_1}^{m_2} (b_i - a_i) \\ &= \sum_{i=1}^n (b_i - a_i) + B - A + b_{m_2} - B - (b_{m_2} - a_{m_2}) \\ &\quad + A - a_{m_1} - (b_{m_1} - a_{m_1}) - \sum_{i=m_1+1}^{m_2-1} (b_i - a_i) \\ &= \sum_{i=1}^n (b_i - a_i) + B - A - (B - a_{m_2}) - (b_{m_1} - A) - \sum_{i=m_1+1}^{m_2-1} (b_i - a_i) \\ &= m(U) + m(V) - m(U \cap V) \end{aligned}$$

For the last “=” note that

$$U \cap V = (A, b_{m_1}) \cup (a_{m_2}, B) \cup \bigcup_{i=m_1+1}^{m_2-1} (a_i, b_i).$$

There are the following other 7 cases, all of which can be shown in a similar way and are left as an exercise.

Case 2:  $a_{m_1} \leq A < B \leq b_{m_2}$  and  $m_1 = m_2$ .

Case 3:  $A < a_{m_1} < B \leq b_{m_2}$  and  $m_1 < m_2$ .

Case 4:  $A < a_{m_1} < B \leq b_{m_2}$  and  $m_1 = m_2$ .

Case 5:  $a_{m_1} \leq A < b_{m_2} < B$  and  $m_1 < m_2$ .

Case 6:  $a_{m_1} \leq A < b_{m_2} < B$  and  $m_1 = m_2$ .

Case 7:  $A < a_{m_1} < b_{m_2} < B$  and  $m_1 < m_2$ .

Case 8:  $A < a_{m_1} < b_{m_2} < B$  and  $m_1 = m_2$ .

Now let  $V = \bigcup_{i=1}^m (c_i, d_i)$  with  $(c_i, d_i)$  being pairwise disjoint for  $i = 1, \dots, m$ .

We will show by induction for all  $m \in \mathbb{N}$  that  $m(U \cup V) = m(U) + m(V) - m(U \cap V)$ .

If  $m = 1$ , thus  $V$  is an open interval, we already have shown our claim.

Assume the claim to be true for  $m - 1$ . Then

$$\begin{aligned}
m(U \cup V) &= m\left(\bigcup_{i=1}^n (a_i, b_i) \cup \bigcup_{i=1}^m (c_i, d_i)\right) \\
&= m(U' \cup (c_m, d_m)) \\
&\quad \left[ \text{where we define } U' = \bigcup_{i=1}^n (a_i, b_i) \cup \bigcup_{i=1}^{m-1} (c_i, d_i) \right] \\
&= m(U') + d_m - c_m - m(U' \cap (c_m, d_m)) \\
&\quad [\text{follows from case } m = 1] \\
&= m\left(U \cup \bigcup_{i=1}^{m-1} (c_i, d_i)\right) + d_m - c_m - m(U \cap (c_m, d_m)) \\
&\quad [\text{Note that } U' \cap (c_m, d_m) = U \cap (c_m, d_m)] \\
&= m(U) + m\left(\bigcup_{i=1}^{m-1} (c_i, d_i)\right) - m\left(U \cap \bigcup_{i=1}^{m-1} (c_i, d_i)\right) \\
&\quad + d_m - c_m - m(U \cap (c_m, d_m)) \\
&\quad [\text{Follows from induction hypothesis}] \\
&= m(U) + m\left(\bigcup_{i=1}^{m-1} (c_i, d_i)\right) + d_m - c_m \\
&\quad - \left[ m\left(U \cap \bigcup_{i=1}^{m-1} (c_i, d_i)\right) + m(U \cap (c_m, d_m)) \right] \\
&= m(U) + m(V) - m(U \cap V) \\
&\quad \left[ \text{Note that } U \cap \bigcup_{i=1}^{m-1} (c_i, d_i) \text{ and } U \cap (c_m, d_m) \text{ are sets} \right. \\
&\quad \left. \text{to which you can apply part (b)} \right]
\end{aligned}$$

which finishes the proof of the induction step.  $\square$

**Exercise 2.** Show that  $\mathcal{F}$  (we defined it to be the sets consisting of finite unions of open intervals) is closed under taking finite intersections as well as closed under taking finite unions.

**Exercise 3.** Prove part (a) and (b) of Proposition 2.

**Exercise 4.** Prove that for  $U \in \mathcal{F}$  and  $V = (A, B)$  it follows that

$$m(U \cup V) = m(U) + m(V) - m(U \cap V),$$

for all the seven cases left in the proof of part (c) of Proposition 2.

**Exercise 5.** Show (by induction for all  $n \in \mathbb{N}$ ) that if  $U_1, U_2, \dots, U_n$  are in  $\mathcal{F}$ , then

$$m\left(\bigcup_{i=1}^n U_i\right) \leq \sum_{i=1}^n m(U_i).$$

In our next step we want to define the measure of arbitrary open sets. First we will need the following theorem.

**Theorem 3.** For every open set  $U \subset \mathbb{R}$  there is a finite or infinite sequence  $(I_\ell)$  of pairwise disjoint, non-empty open intervals so that

$$U = \bigcup_{\ell=1}^n I_\ell \quad (\text{if sequence has length } n) \quad \text{or}$$

$$U = \bigcup_{\ell=1}^{\infty} I_\ell \quad (\text{if sequence has infinite length}).$$

This representation is unique in the following sense:

Either  $U \in \mathcal{F}$ , then  $U$  cannot be the union of infinitely many disjoint non-empty open intervals, and the representation as a finite union of disjoint non-empty open intervals is unique by Lemma 1.

Or  $U \notin \mathcal{F}$ , then  $U$  is the union of infinitely countably many disjoint non-empty open intervals, and if

$$U = \bigcup_{\ell=1}^{\infty} I_\ell \quad \text{and} \quad U = \bigcup_{\ell=1}^{\infty} \tilde{I}_\ell,$$

where  $(I_\ell)$  and  $(\tilde{I}_\ell)$  are both sequences of disjoint non empty open intervals, then there is a bijection:

$$\sigma : \mathbb{N} \rightarrow \mathbb{N} \quad \text{so that} \quad I_\ell = \tilde{I}_{\sigma(\ell)}, \quad \text{for } \ell \in \mathbb{N}.$$

*Proof.* Let  $U$  be an open set. Put  $D = U \cap \mathbb{Q}$ . Since  $\mathbb{Q}$  is countable,  $D$  is countable and we can write it as a sequence  $D = \{d_n : n \in \mathbb{N}\}$ .

Consider  $d_1$ . And define  $\mathcal{I}_1$  to be the set of all open intervals  $I$  which contain  $d_1$  as an element and are subsets of  $U$ . Since  $U$  is open,  $\mathcal{I}_1$  is not empty. Then we define:

$$I_1 = \bigcup \{I : I \in \mathcal{I}_1\}.$$

First note that  $I_1$  is an interval. Indeed, if  $x < y$  are both in  $I_1$ , there must be an  $I_x \in \mathcal{I}_1$  and an  $I_y \in \mathcal{I}_1$  so that  $x \in I_x$  and  $y \in I_y$ . Since  $I_x$  and  $I_y$  have the point  $d_1$  in common, it follows that  $I_x \cup I_y$  is an interval in  $\mathcal{I}_1$  containing  $x, y$  thus  $[x, y] \subset I_x \cup I_y \subset I_1$ .

Secondly,  $I_1$  is the union of open intervals and must be therefore open.

Writing now  $I_1$  as  $I_1 = (a_1, b_1)$  we want to show that the endpoints  $a_1$  and  $b_1$  cannot be in  $U$ . Indeed, assume that for example  $a_1 \in U$ . Then for some  $\varepsilon > 0$  we have  $(a_1 - \varepsilon, a_1 + \varepsilon) \subset U$ . Therefore it follows that  $I_1 \cup (a_1 - \varepsilon, a_1 + \varepsilon) = (a_1 - \varepsilon, b_1)$  is an open interval which contains  $d_1$  and

is contained in  $U$ . We conclude that  $I_1 \cup (a_1 - \varepsilon, a_1 + \varepsilon) = (a_1 - \varepsilon, b_1)$  is an element of  $\mathcal{I}_1$  which is a contradiction since  $I_1$  was defined to be the union of all elements of  $\mathcal{I}_1$ .

Therefore we can write:

$$U = (a_1, b_1) \cup [U \setminus (a_1, b_1)] = (a_1, b_1) \cup [U \setminus [a_1, b_1]].$$

This means that  $U_2 = U \setminus [a_1, b_1]$  is open. Either  $U_2$  is empty, meaning that  $U = (a_1, b_1)$ , and we are finished. Or  $U_2$  is not empty and since it is open, the set  $D_2 = D \cap U_2 = \mathbb{Q} \cap U_2$  is dense in  $U_2$  and we can define  $n_2 = \min\{n \in \mathbb{N} : d_n \in U_2\}$ .

Now we repeat the procedure for  $d_{n_2}$  instead of  $d_1$  and  $U_2$  instead of  $U$ . Thus, find an open non-empty interval  $(a_2, b_2)$  so that  $d_{n_2} \in (a_2, b_2) \subset U_2 \subset U$  so that  $a_2, b_2 \notin U_2$ . This implies that  $a_2, b_2 \notin U$  (otherwise we would have, say  $a_2 \in U$  and thus  $a_2 \in (a_1, b_1)$ , which would imply that  $(a_2, b_2) \cap (a_1, b_1) \neq \emptyset$ ).

We can continue this way and find numbers  $1 < n_2 < n_3 < \dots$ , open sets  $U \supset U_2 \supset U_3 \dots$ , and open disjoint intervals  $(a_1, b_1) \subset U$ ,  $(a_2, b_2) \subset U_2$ ,  $(a_3, b_3) \subset U_3$  and positive integers  $n_1 = 1 < n_2 < n_3 < \dots$  so that:

- (1) Either for some  $i_0 \in \mathbb{N}$  the set  $U_{i_0}$  is empty, then we stop the choice of  $n_i$ 's and  $(a_i, b_i)$ 's and  $U_i$ 's, otherwise put  $i_0 = \infty$ .
- (2)  $(a_i, b_i)_{i < i_0}$  are pairwise disjoint,
- (3)  $a_i, b_i \notin U$ , for  $i < i_0$ ,
- (4)  $U_{i+1} = U \setminus [\bigcup_{j=1}^i (a_j, b_j)]$ , and  $(a_i, b_i) \subset U_i \subset U$ , for  $i < i_0$ ,
- (5)  $n_1 = 1$ , and  $n_{i+1} = \min\{n : d_n \notin \bigcup_{j=1}^i (a_j, b_j)\}$ , if  $1 < i < i_0$ , and  $d_{n_i} \in (a_i, b_i)$ , for  $i < i_0$

Either for some  $m \in \mathbb{N}$  we get that  $U_m = \emptyset$  (meaning  $i_0 < \infty$ ). Then we deduce that

$$\begin{aligned} U &= (a_1, b_1) \cup U_2 \\ &= (a_1, b_1) \cup (a_2, b_2) \cup U_3 \\ &\vdots \\ &= \dots (a_1, b_1) \cup (a_2, b_2) \dots (a_{m_1}, b_{m_1}). \end{aligned}$$

Or the procedure continues forever (meaning  $i_0 = \infty$ ) and we get an infinite sequence  $(a_i, b_i)_{i \in \mathbb{N}}$ . From the construction, it is clear that the  $(a_i, b_i)$  are disjoint and that  $\bigcup_{i=1}^{\infty} (a_i, b_i) \subset U$ .

Let us prove that  $U \subset \bigcup_{i=1}^{\infty} (a_i, b_i)$  and let therefore  $x \in U$ . Since  $U$  is open there is an  $\varepsilon > 0$  so that  $(x - \varepsilon, x + \varepsilon) \subset U$  and there must be an  $n \in \mathbb{N}$  so that  $d_n \in (x, x + \varepsilon)$ . For this  $n$  there must be an  $i \in \mathbb{N}$  so that  $n_{i-1} \leq n < n_i$ . This means that  $d_n$  must be in some interval  $(a_\ell, b_\ell)$  with  $\ell \leq i - 1$ . Now since  $a_\ell, b_\ell \notin U$ , but  $[x, d_n] \subset U$ , as well as  $(a_\ell, d_n] \subset U$ , and  $[d_n, b_\ell) \subset U$ ,  $a_\ell$  and  $b_\ell$  cannot lie between  $x$  and  $d_n$  which implies that  $x \in [x, d_n] \subset (a_\ell, b_\ell)$  (draw a picture).

Now we show that the representation of  $U$  as countable union of pairwise disjoint open and non-empty intervals is unique .

Assume

$$\bigcup_{i \in I} (a_i, b_i) = \bigcup_{j \in J} (c_j, d_j),$$

where  $((a_i, b_i))_{i \in I}$  and  $((c_j, d_j))_{j \in J}$  are both countable (finite or infinite) families of pairwise disjoint open intervals.

We have to find a bijection  $\sigma : I \rightarrow J$  so that  $(a_i, b_i) = (c_{\sigma(i)}, d_{\sigma(i)})$ .

So let  $i \in I$ , pick an arbitrary element  $x \in (a_i, b_i)$ , say the midpoint of the interval. Since  $x \in \bigcup_{j \in J} (c_j, d_j)$  there must be a (unique)  $j \in J$  so that  $x \in (c_j, d_j)$ . Define  $\sigma(i) = j$ .

First we claim that  $a_i = c_{\sigma(i)}$  and  $b_i = d_{\sigma(i)}$ .

Assume for example that  $a_i < c_{\sigma(i)}$ . Since  $x \in (c_{\sigma(i)}, d_{\sigma(i)})$  it follows that  $a_i < c_{\sigma(i)} < x < b_i$ , therefore  $c_{\sigma(i)} \in (a_i, b_i)$ , which means that

$$c_{\sigma(i)} \in \bigcup_{\ell \in I} (a_\ell, b_\ell) = \bigcup_{m \in J} (c_m, d_m).$$

This means that there must be an  $m \in J$  so that  $c_{\sigma(i)} \in (c_m, d_m)$ . Of course  $m \neq \sigma(i)$ , and, thus there must be an  $\varepsilon > 0$  so that  $(c_{\sigma(i)} - \varepsilon, c_{\sigma(i)} + \varepsilon) \subset (c_m, d_m)$ . But this implies that  $c_{\sigma(i)} + \varepsilon/2 \in (c_m, d_m) \cap (c_{\sigma(i)}, d_{\sigma(i)})$ , which is a contradiction since  $(c_m, d_m)$  and  $(c_{\sigma(i)}, d_{\sigma(i)})$  were assumed to be disjoint.

With the same arguments we conclude that  $a_i > c_{\sigma(i)}$  is impossible. Thus we have shown that  $a_i = c_{\sigma(i)}$ .

The proof that  $b_i = d_{\sigma(i)}$  follows the same lines. It is left to show that  $\sigma : I \rightarrow J$  is a bijection.

If  $\sigma(i) = \sigma(i')$  then it follows from what we just proved, that

$$(a_i, b_i) = (c_{\sigma(i)}, d_{\sigma(i)}) = (c_{\sigma(i')}, d_{\sigma(i')}) = (a_{i'}, b_{i'}),$$

which implies that  $i = i'$ , and, thus, that  $\sigma(\cdot)$  is injective.

In order to show that  $\sigma(\cdot)$  is surjective let  $j_0 \in J$ . Assume there is no  $i \in I$  so that  $j_0 = \sigma(i)$ . But then we would have that

$$\bigcup_{i \in I} (a_i, b_i) = \bigcup_{i \in I} (c_{\sigma(i)}, d_{\sigma(i)}) \subsetneq (c_{j_0}, d_{j_0}) \cup \bigcup_{i \in I} (c_{\sigma(i)}, d_{\sigma(i)}) \subset \bigcup_{j \in J} (c_j, d_j),$$

which is a contradiction to our assumption and finishes the proof.  $\square$

**Exercise 6.** (filling the gaps of proof of Theorem 3).

Assume

$$\bigcup_{i \in I} (a_i, b_i) = \bigcup_{j \in J} (c_j, d_j),$$

where  $((a_i, b_i))_{i \in I}$  and  $((c_j, d_j))_{j \in J}$  are both countable (finite or infinite) families of pairwise disjoint open intervals.

Define  $\sigma : I \rightarrow J$  as defined in the proof of Theorem 3. Show that  $b_i = d_{\sigma(i)}$ .

With the help of Theorem 3 we are in the position to define what we mean by the *measure of open sets*.

Let  $U$  be open and  $\{J_\ell : \ell = 1, 2, \dots, n\}$  or  $\{J_\ell : \ell \in \mathbb{N}\}$  the (unique) finite or infinite set of pairwise disjoint, non-empty, open intervals whose union is  $U$ .

Then we put

$$m(U) = \sum_{\ell=1}^n m(J_\ell) \text{ or } m(U) = \sum_{\ell=1}^{\infty} m(J_\ell) = \lim_{N \rightarrow \infty} \sum_{\ell=1}^N m(J_\ell), \text{ respectively.}$$

**Remark.** Note that we could have  $m(U) = \infty$ .

The facts observed in Proposition 2 still hold true.

**Proposition 4.** *The map*

$$m : \{U \subset \mathbb{R} : U \text{ open}\} \rightarrow [0, \infty]$$

*has the following properties.*

- a)  $m$  is monotone: If  $U, V \subset \mathbb{R}$  open and  $U \subset V$  then,  $m(U) \leq m(V)$ .
- b)  $m$  is additive on disjoint sets: If  $U, V \subset \mathbb{R}$  open and  $U \cap V = \emptyset$  then  $m(U \cup V) = m(U) + m(V)$ .

*More generally, if  $U_1, U_2, \dots, U_n$  are pairwise disjoint and open then*

$$m\left(\bigcup_{\ell=1}^n U_\ell\right) = \sum_{\ell=1}^n m(U_\ell).$$

- c) If  $U, V \subset \mathbb{R}$  open and  $m(U \cap V) < \infty$  (otherwise equation below not welldefined)

$$m(U \cup V) = m(U) + m(V) - m(U \cap V)$$

*In particular  $m$  is subadditive:  $m(U \cup V) \leq m(U) + m(V)$ .*

*More generally, if  $U_1, U_2, \dots, U_n$  are open then*

$$m\left(\bigcup_{\ell=1}^n U_\ell\right) \leq \sum_{\ell=1}^n m(U_\ell).$$

- d) Assume that  $(U_n)$  is a sequence of open sets with  $U_1 \subset U_2 \subset U_3 \dots$  and let  $U = \bigcup_{n=1}^{\infty} U_n$  then

$$m(U) = \lim_{n \rightarrow \infty} m(U_n).$$

*Proof.* We may assume that  $U$  and  $V$  are the countable infinite union of open disjoint intervals  $(I_\ell)$  and  $(J_\ell)$  respectively,

$$U = \bigcup_{\ell=1}^{\infty} I_\ell \text{ and } V = \bigcup_{\ell=1}^{\infty} J_\ell.$$

Note that the finite case is included if we allow the  $I_\ell$ 's and  $J_\ell$ 's to be empty.

For  $\ell \in \mathbb{N}$ , let  $a_\ell \leq b_\ell$  and  $c_\ell \leq d_\ell$  be such that  $I_\ell = (a_\ell, b_\ell)$  and  $J_\ell = (c_\ell, d_\ell)$ .

In order to show (a), assume that  $U \subset V$ . Since  $m(U) = \lim_{N \rightarrow \infty} \sum_{\ell=1}^N m(I_\ell)$  we have to show for an arbitrary  $N$  that  $m(\bigcup_{\ell=1}^N I_\ell) \leq m(V)$ . We will use a *compactness trick* to achieve that.

Let  $\varepsilon > 0$  arbitrary. Then we can choose for  $\ell \leq N$  two numbers  $\tilde{a}_\ell, \tilde{b}_\ell$  in  $(a_\ell, b_\ell)$  so that

$$\tilde{a}_\ell - a_\ell < \varepsilon/2N \text{ and } b_\ell - \tilde{b}_\ell < \varepsilon/2N.$$

Note that it follows that

$$\bigcup_{\ell=1}^N (\tilde{a}_\ell, \tilde{b}_\ell) \subset \bigcup_{\ell=1}^N [\tilde{a}_\ell, \tilde{b}_\ell] \subset \bigcup_{\ell=1}^N (a_\ell, b_\ell),$$

and, thus,

$$\begin{aligned} m\left(\bigcup_{\ell=1}^N (\tilde{a}_\ell, \tilde{b}_\ell)\right) &\leq m\left(\bigcup_{\ell=1}^N (a_\ell, b_\ell)\right) \\ &= \sum_{\ell=1}^N b_\ell - a_\ell \\ &\leq \varepsilon + \sum_{\ell=1}^N \tilde{b}_\ell - \tilde{a}_\ell = \varepsilon + m\left(\bigcup_{\ell=1}^N (\tilde{a}_\ell, \tilde{b}_\ell)\right). \end{aligned}$$

Now we let  $K = \bigcup_{\ell=1}^N [\tilde{a}_\ell, \tilde{b}_\ell]$ , which is a compact set, and observe that

$$K \subset \bigcup_{\ell=1}^N (a_\ell, b_\ell) \subset U \subset V = \bigcup_{\ell=1}^{\infty} J_\ell.$$

Since every open covering of a compact set has a finite subcover we can find an  $M \in \mathbb{N}$  so that

$$\bigcup_{\ell=1}^N (\tilde{a}_\ell, \tilde{b}_\ell) \subset K \subset \bigcup_{\ell=1}^M J_\ell.$$

This implies by Proposition 2

$$m(V) \geq m\left(\bigcup_{\ell=1}^M J_\ell\right) \geq m\left(\bigcup_{\ell=1}^N (\tilde{a}_\ell, \tilde{b}_\ell)\right) \geq m\left(\bigcup_{\ell=1}^N (a_\ell, b_\ell)\right) - \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary this implies that

$$m(V) \geq m\left(\bigcup_{\ell=1}^N (a_\ell, b_\ell)\right),$$

since  $N \in \mathbb{N}$  is arbitrary,

$$m(V) \geq \lim_{N \rightarrow \infty} m \left( \bigcup_{\ell=1}^N (a_\ell, b_\ell) \right) = m(U).$$

In order to show (b), we assume that  $U$  and  $V$  are disjoint, then  $U \cup V = \bigcup_{\ell=1}^{\infty} I_\ell \cup \bigcup_{\ell=1}^{\infty} J_\ell$ . Since  $I_\ell \cap J_m = \emptyset$  for all  $\ell, m \in \mathbb{N}$  and since the representation of  $U \cup V$  as a union of a countable set of disjoint open, and non-empty intervals is unique, we conclude that, the non-empty elements of  $\{I_\ell : \ell \in \mathbb{N}\} \cup \{J_\ell : \ell \in \mathbb{N}\}$  must be *the unique* set of pairwise disjoint, open and non-empty intervals, whose union is  $U \cup V$ . Thus, by definition of  $m(U \cup V)$ ,

$$m(U \cup V) = \sum_{\ell \in \mathbb{N}} m(I_\ell) + \sum_{\ell \in \mathbb{N}} m(J_\ell) = m(U) + m(V).$$

It follows now easily by induction for all  $n \in \mathbb{N}$  that  $U_1, U_2, \dots, U_n$  are pairwise disjoint open sets it follows that

$$m \left( \bigcup_{\ell=1}^n U_\ell \right) = \sum_{\ell=1}^n m(U_\ell).$$

We will show (d) now, which will follow from a similar compactness argument we used for (a). Let  $U = \bigcup U_n$  where  $U_n$  is an increasing sequence of open sets. Since  $U$  is also open we can write it as a countable union  $U = \bigcup I_\ell$  where the  $I_\ell$ 's are pairwise disjoint and open.

First, since  $U_n \subset U$  it follows from part (a) that  $m(U_n) \leq m(U)$  for all  $n \in \mathbb{N}$  and thus

$$\lim_{n \rightarrow \infty} m(U_n) \leq m(U).$$

To show the reversed inequality we have to distinguish between two cases. Case 1:  $m(U) < \infty$ . Let  $\varepsilon > 0$  arbitrary then there is an  $N \in \mathbb{N}$  so that

$$m \left( \bigcup_{\ell=1}^N I_\ell \right) \geq m(U) - \varepsilon/2.$$

As in above compactness argument we make the intervals  $I_\ell$  a *little bit smaller*, i.e. we choose open intervals  $\tilde{I}_\ell \subset I_\ell$  so that

$$\tilde{I}_\ell \subset \overline{\tilde{I}_\ell} \subset I_\ell, \text{ and } m(\tilde{I}_\ell) \geq m(I_\ell) - \varepsilon/4N.$$

( $\overline{\tilde{I}_\ell}$  denotes the closure of  $\tilde{I}_\ell$ , i.e. the interval which includes the endpoints.)

Since

$$\bigcup_{\ell=1}^N \overline{\tilde{I}_\ell} \subset U = \bigcup_{\ell=1}^{\infty} U_n,$$

it follows from the compactness of the set

$$K = \bigcup_{\ell=1}^N \overline{\tilde{I}_\ell}$$

that there is an  $M \in \mathbb{N}$  so that

$$\bigcup_{\ell=1}^N \tilde{I}_\ell \subset K \subset U_M.$$

Thus, we have the following chain of inequalities:

$$\begin{aligned} \lim_{n \rightarrow \infty} m(U_n) &\geq m(U_M) \geq m\left(\bigcup_{\ell=1}^N \tilde{I}_\ell\right) \\ &= \sum_{\ell=1}^N m(\tilde{I}_\ell) \geq \sum_{\ell=1}^N m(I_\ell) - \varepsilon/2 \geq m(U) - \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary we deduce that  $\lim_{n \rightarrow \infty} m(U_n) \geq m(U)$ .

Case 2:  $m(U) = \infty$ . In this case we need to show that  $\lim_{n \rightarrow \infty} m(U_n) = \infty$ .

Thus, given a number  $C > 0$ , we need to show that there is an  $N$  so that  $m(U_N) \geq C$ .

Since

$$\infty = m(U) = \lim_{M \rightarrow \infty} \sum_{\ell=1}^M m(I_\ell) = \lim_{M \rightarrow \infty} m\left(\bigcup_{\ell=1}^M I_\ell\right),$$

we can find an  $M \in \mathbb{N}$  so that

$$m\left(\bigcup_{\ell=1}^M I_\ell\right) \geq 2C.$$

Again, we make the intervals  $I_\ell$ ,  $\ell \leq M$  a *little bit smaller*, i.e. we choose open intervals  $\tilde{I}_\ell \subset I_\ell$  so that

$$\tilde{I}_\ell \subset \overline{\tilde{I}_\ell} \subset I_\ell, \text{ and } m(\tilde{I}_\ell) \geq m(I_\ell) - C/M.$$

It follows again from the compactness of the set

$$K = \bigcup_{\ell=1}^M \overline{\tilde{I}_\ell}$$

that there is an  $N \in \mathbb{N}$  so that

$$\bigcup_{\ell=1}^M \tilde{I}_\ell \subset K \subset U_N.$$

Now it follows that

$$m(U_N) \geq m\left(\bigcup_{\ell=1}^M \tilde{I}_\ell\right) = \sum_{\ell=1}^M m(\tilde{I}_\ell) \geq -C + \sum_{\ell=1}^M m(I_\ell) = -C + m\left(\bigcup_{\ell=1}^M I_\ell\right) \geq C,$$

which implies that  $\lim_{n \rightarrow \infty} m(U_n) = \infty$ , since  $C > 0$  was arbitrary.

Finally we want to show (c). Let  $U$  and  $V$  be the countable infinite union of open disjoint intervals  $(I_\ell)$  and  $(J_\ell)$  respectively. Applying part (d) (which we have already shown) to  $W_n = \bigcup_{\ell=1}^n I_\ell \cup J_\ell$  we deduce that

$$m(U \cup V) = m\left(\bigcup_n W_n\right) = \lim_{n \rightarrow \infty} m(W_n).$$

Secondly we can apply Proposition 2 part (c), to  $U_n = \bigcup_{\ell=1}^n I_\ell$  and  $V_n = \bigcup_{\ell=1}^n J_\ell$  and taking a limit we obtain

$$\begin{aligned} m(U \cup V) &= \lim_{n \rightarrow \infty} m(W_n) \\ &= \lim_{n \rightarrow \infty} m(U_n) + m(V_n) - m(U_n \cap V_n) \\ &= m(U) + m(V) - m(U \cap V). \end{aligned}$$

For the last “=” note that  $U_n$  increases to  $U$ ,  $V_n$  increases to  $V$ , and  $U_n \cap V_n$  increases to  $U \cap V$ .

The part of (c) which starts by “more generally” is left as an exercise.  $\square$

**Exercise 7.** *Prove the part which starts by “more generally” in Proposition 4 part (c).*

In the next step we want to define the measure of compact sets. We will now consider a bounded interval of the form  $[-N, N]$ ,  $N > 0$ , and only consider subsets of  $[-N, N]$ .

Recall that a set  $V \subset [-N, N]$  is open in  $[-N, N]$  if there is an open set  $\tilde{V} \subset \mathbb{R}$  so that  $V = \tilde{V} \cap [-N, N]$ . For example, the set  $[-N, N/2)$  is open in  $[-N, N]$ . By Theorem 3 we can write  $\tilde{V}$  as a (finite or infinite) countable union of open intervals, say  $\tilde{V} = \bigcup \tilde{I}_j$ . Letting now  $I_j = \tilde{I}_j \cap [-N, N]$ , it follows that  $I_j$  is either of the form  $(a_j, b_j)$ , of the form  $[-N, b_j)$ , of the form  $(a_j, N]$  or  $[-N, N]$ . It follows that  $V = \bigcup I_j$ , and that  $(I_j)$  is an (infinite or finite) sequence of disjoint open intervals in  $[-N, N]$ . Therefore we can define

$$(3) \quad m(V) = \sum_{\ell=1}^n m(I_\ell) \text{ or } m(V) = \sum_{\ell=1}^{\infty} m(I_\ell), \text{ respectively.}$$

**Exercise 8.** *Prove that all the claims of Proposition 4 hold if one replaces “open” by “open in  $[-N, N]$ ” and show that for all sets  $U$  which are open in  $[-N, N]$  it follows that  $m(U) \leq 2N$ .*

**Hint:** *First observe that if  $U$  is open in  $[-N, N]$ , then you can make out of  $U$  a set which is open in  $\mathbb{R}$  by taking at most two points away.*

If  $K \subset [-N, N]$  is closed, and therefore  $V = [-N, N] \setminus K$  is open in  $[-N, N]$ , we define

$$(4) \quad m(K) = 2N - m([-N, N] \setminus K)$$

**Exercise 9.** We want to show that the definition of  $m(K)$  for  $K$  does not depend on  $N$ .

Assume that  $N' > N$  and that  $K \subset [-N, N]$  is compact. Then it follows that

$$2N - m([-N, N] \setminus K) = 2N' - m([-N', N'] \setminus K).$$

**Proposition 5.**

a) If  $K = \bigcup_{i=1}^{\ell} [a_i, b_i]$ , with  $-N \leq a_1 \leq b_1 \leq a_2 \leq \dots \leq a_n \leq b_n \leq N$ , then

$$m(K) = \sum_{i=1}^{\ell} b_i - a_i.$$

b)  $m$  is monotone on the set of all compact subsets of  $[-N, N]$ : If  $K \subset \tilde{K} \subset [-N, N]$  are compact, then  $m(K) \leq m(\tilde{K})$ .

c) If  $(K_n)$  is a decreasing sequence of compact subsets then

$$m\left(\bigcap_{n \in \mathbb{N}} K_n\right) = \lim_{n \rightarrow \infty} m(K_n).$$

d) If  $K \subset [-N, N]$  is compact and  $U$  is open in  $[-N, N]$ , and  $K \subset U$  then  $m(K) \leq m(U)$ .

*Proof.* (a), (b) and (c) are easy and left as an exercise. To proof (d) assume that  $K \subset [-N, N]$  is compact,  $U \subset [-N, N]$  is open in  $[-N, N]$  and  $K \subset U$ . by Theorem 3 we can write  $U$  as

$$U = \bigcup_{\ell \in \mathbb{N}} (a_{\ell}, b_{\ell}),$$

where the  $(a_{\ell}, b_{\ell})$ 's are disjoint (in case that  $a_{\ell} = -N$  the interval  $(a_{\ell}, b_{\ell})$  could be replaced by  $[a_{\ell}, b_{\ell})$  and in case that  $b_{\ell} = N$   $(a_{\ell}, b_{\ell})$  could be replaced by  $(a_{\ell}, N]$ , but disregard this possibilities, noting that the arguments would not change). By compactness of  $K$  we find an  $\ell \in \mathbb{N}$  so that

$$K \subset \bigcup_{i=1}^{\ell} (a_i, b_i).$$

This means that

$$[-N, N] \setminus K \supset [-N, a_1] \cup [b_1, a_2] \cup [b_2, a_3] \cup \dots \cup [b_{\ell}, N],$$

and since  $[-N, N] \setminus K$  is open in  $[-N, N]$  it follows that

$$[-N, N] \setminus K \supset [-N, a_1) \cup (b_1, a_2) \cup (b_2, a_3) \cup \dots \cup (b_{\ell}, N],$$

Now we can use the monotonicity of  $m$  for open sets (by Proposition 4 and Exercises 7) and obtain that

$$m([-N, N] \setminus K) \geq (a_1 - (-N)) + (a_2 - b_1) + (a_3 - b_2) + \dots + (a_{\ell} - b_{\ell-1}) + (N - b_{\ell}).$$

Therefore

$$\begin{aligned}
 m(K) &= 2N - m([-N, N] \setminus K) \\
 &\leq 2N - [(a_1 - (-N)) + (a_2 - b_1) + (a_3 - b_2) + \\
 &\quad \dots + (a_\ell - b_{\ell-1}) + (N - b_\ell)] \\
 &= b_1 - a_1 + b_2 - a_2 + \dots + b_\ell - a_\ell \\
 &= m\left(\bigcup_{i=1}^{\ell} (a_i, b_i)\right) \leq m(U),
 \end{aligned}$$

which finishes the proof (d). □

**Exercise 10.** Prove part (a), (b) and (c) of Proposition 5

**We are now coming to the crucial Definition.**

**Definition 6.** We call a subset  $A \subset [-N, N]$  measurable in  $[-N, N]$  if the following holds:

(\*)  $\forall \varepsilon > 0 \exists K \subset A, K \text{ compact } \exists U \supset A, U \text{ open in } [-N, N] \text{ so that:}$

$$m(U) - m(K) < \varepsilon$$

If  $A$  is measurable in  $[-N, N]$  we define:

(5)  $m(A) = \inf\{m(U) : U \text{ open in } [-N, N] \text{ and } A \subset U\}.$

Before we can go, on we have to observe that we did not redefine the measure of sets, we have already defined before, to be something different. More precisely we need to show that for open or compact  $A$  5 coincides with the previous definition of  $m(A)$

Let us for the moment define

$$\tilde{m}(A) = \inf\{m(U) : U \text{ open in } [-N, N] \text{ and } A \subset U\}.$$

We will have to show that open and compact sets in  $[-N, N]$  are measurable and that for open sets  $U$  and compact sets  $K$  we have  $\tilde{m}(K) = m(K)$ ,  $\tilde{m}(U) = m(U)$ . Then we do not need the “ $\sim$ ” anymore.

**Proposition 7.** Open sets and compact sets in  $[-N, N]$  are measurable and if  $U$  is open and  $K$  is compact in  $[-N, N]$  then

$$\tilde{m}(K) = m(K) \quad \text{and} \quad \tilde{m}(U) = m(U).$$

*Proof.* We first need to show that open sets in  $[-N, N]$  are measurable.

Assume  $U$  is open. Then, by Theorem 3, we can write

$$U = \bigcup_{i \in I} (a_i, b_i) \cup [-N, a] \cup (b, N],$$

Where either  $I = \mathbb{N}$  or  $I = \{1, 2, \dots, n\}$  for some  $n \in \mathbb{N}$ , and where  $a = -N$  if  $-N \notin U$ , and  $b = N$  if  $N \notin U$  and where the  $(a_i, b_i)$ 's are pairwise disjoint subsets of  $(a, b)$ . If  $I = \{1, 2, \dots, n\}$  is finite choose  $n_0 = n$ , otherwise choose  $n_0$  large enough so that (recall that  $m(U)$  is finite)

$$m(U) = \lim_{m \rightarrow \infty} \sum_{i=1}^m b_i - a_i < \sum_{i=1}^{n_0} b_i - a_i + \varepsilon/2.$$

Then we choose for  $\ell \leq n_0$  numbers  $\tilde{a}_\ell$  and  $\tilde{b}_\ell$  so that  $a_\ell < \tilde{a}_\ell < \tilde{b}_\ell < b_\ell$  and so that  $b_\ell - \tilde{b}_\ell < \varepsilon/4n_0$  and  $\tilde{a}_\ell - a_\ell < \varepsilon/4n_0$ . We define the compact set  $K = \bigcup_{\ell=1}^{n_0} [\tilde{a}_\ell, \tilde{b}_\ell]$  which, by Proposition 5 part (a), we have

$$m(K) = \sum_{\ell=1}^{n_0} \tilde{b}_\ell - \tilde{a}_\ell < \varepsilon/2 + \sum_{\ell=1}^{n_0} b_\ell - a_\ell < \varepsilon + m(U),$$

thus for the compact set  $K$  it follows that  $U$  lies between  $K$  and itself and  $m(U) - m(K) < \varepsilon$ .

Secondly, from the monotonicity of  $m(\cdot)$  on open sets, it follows that,

$$m(U) = \inf\{m(V) : V \text{ open in } [-N, N], U \subset V\} = \tilde{m}(U).$$

Now let  $K$  be compact, and let  $\varepsilon > 0$ . Then  $[-N, N] \setminus K$  is open in  $[-N, N]$  and we just proved that this implies that  $[-N, N] \setminus K$  is measurable. We therefore find a compact set  $C \subset [-N, N] \setminus K$  so that  $m([-N, N] \setminus K) - m(C) < \varepsilon$ .

Therefore it follows that  $K \subset [-N, N] \setminus C$ ,  $[-N, N] \setminus C$  is open in  $[-N, N]$  and

$$\begin{aligned} m([-N, N] \setminus C) - m(K) &= 2N - m(C) - m(K) \\ &\quad [\text{by definition of } m(C), \text{ for } C \text{ compact}] \\ &= 2N - m(C) - (2N - m([-N, N] \setminus K)) \\ &\quad [\text{again, by definition of } m(K), \text{ for } K \text{ compact}] \\ &= m([-N, N] \setminus K) - m(C) < \varepsilon, \end{aligned}$$

which implies that  $K$  is measurable in  $[-N, N]$ , since we  $\varepsilon > 0$  is arbitrary. From Proposition 5 it follows that

$$\tilde{m}(K) = \inf\{m(V) : V \text{ open in } [-N, N] \text{ and } K \subset V\} \geq m(K).$$

On the other hand we found, for any  $\varepsilon > 0$ , a set  $V$ , namely above defined set  $V = [-N, N] \setminus C$ , which contains  $K$ , is open in  $[-N, N]$  with  $m(V) \leq m(K) + \varepsilon$ . Thus  $\tilde{m}(K) \leq m(K)$ , which finishes our proof.  $\square$

Secondly we want to show that the definition in 5 does not depend on the choice of  $N$ . This means the following: Assume that  $A \subset [-N, N]$  and let  $N' > N$ . The t follows of course that  $A$  is also subset of  $[-N', N']$

**Proposition 8.** *Then  $A$  is measurable as subset of  $[-N, N]$  if and only if it is measurable as subset of  $[-N', N']$ .*

*And in this case*

$$m(A) = \inf\{m(U) : U \text{ open in } [-N, N]\} = \inf\{m(U') : U' \text{ open in } [-N', N']\}.$$

**Exercise 11.** *Prove Proposition 8. First assume that  $A$  is open in  $[-N, N]$ .*

**Exercise 12.** *Show that the following sets are measurable in  $[-N, N]$  and compute their measure:*

- a)  $\emptyset$
- b)  $\{0\}$
- c)  $F \subset [-N, N]$ ,  $F$  finite.
- d)  $(a, b]$  and  $[a, b)$  with  $-N < a < b < N$ .
- e)  $[-N, N] \cap \mathbb{Q}$ .

**Hint for (e):** *For a given  $\varepsilon > 0$  you want to find an open set  $U$  in  $[-N, N]$  which contains all rational numbers of  $[-N, N]$  and has measure smaller than  $\varepsilon$  (sounds like being wrong, doesn't it?).*

**Exercise 13.** *Prove for a set  $A \subset [-N, N]$*

- a)  $A$  is measurable  $\iff [-N, N] \setminus A$  is measurable  
and in that case  $m([-N, N] \setminus A) = 2N - m(A)$ .
- b) If  $A$  is measurable then

$$m(A) = \sup\{m(K) : K \text{ is compact and } K \subset A\}.$$

For the next Theorem we put

$$\mathcal{L}_{[-N, N]} = \{A \subset [-N, N] : A \text{ is measurable}\}.$$

**Theorem 9.** *The set  $\mathcal{L}_{[-N, N]}$  has the following properties:*

- a)  $\emptyset \in \mathcal{L}_{[-N, N]}$
- b)  $\mathcal{L}_{[-N, N]}$  is closed under taking complements:  
$$A \in \mathcal{L}_{[-N, N]} \Rightarrow [-N, N] \setminus A \in \mathcal{L}_{[-N, N]}.$$
- c)  $\mathcal{L}_{[-N, N]}$  is closed under taking countable union: If  $U_n \in \mathcal{L}_{[-N, N]}$ , for  $n \in \mathbb{N}$ , then  $\bigcup_{n \in \mathbb{N}} U_n \in \mathcal{L}_{[-N, N]}$ .

Before we start with the proof of Theorem 9 we need two lemmas.

**Lemma 10.** *For two compact and disjoint sets  $K_1, K_2$  it follows that*

$$m(K_1 \cup K_2) = m(K_1) + m(K_2).$$

*Proof.* Let  $K_1$  and  $K_2$  be disjoint compact subsets of  $[-N, N]$  we first claim that there are two disjoint open sets,  $V_1$  and  $V_2$  in  $[-N, N]$ , with  $K_1 \subset V_1$  and  $K_2 \subset V_2$ . Indeed, for each  $x \in K_1$  there is an  $\varepsilon_x > 0$  so that

$$[-N, N] \cap (x - \varepsilon_x, x + \varepsilon_x) \subset [-N, N] \setminus K_2 \quad ([-N, N] \setminus K_2 \text{ is open in } [-N, N])$$

and for each  $z \in K_2$  there is an  $\varepsilon_z > 0$  so that

$$[-N, N] \cap (z - \varepsilon_z, z + \varepsilon_z) \subset [-N, N] \setminus K_1 \quad ([-N, N] \setminus K_1 \text{ is open in } [-N, N])$$

Choose now “one third the radius”, i.e.

$$V_1 = [-N, N] \cap \bigcup_{x \in K_1} (x - \varepsilon_x/3, x + \varepsilon_x/3) \text{ and}$$

$$V_2 = [-N, N] \cap \bigcup_{z \in K_2} (z - \varepsilon_z/3, z + \varepsilon_z/3).$$

We claim that  $V_1 \cap V_2 = \emptyset$ . Assume that some  $y \in V_1 \cap V_2$ . Then there is an  $x \in K_1$  and a  $z \in K_2$  so that  $y \in (x - \varepsilon_x/3, x + \varepsilon_x/3) \cap (z - \varepsilon_z/3, z + \varepsilon_z/3)$ . This implies that  $|x - z| \leq |x - y| + |y - z| \leq \varepsilon_x/3 + \varepsilon_z/3$ . Let  $\varepsilon_z$  be the smaller of both number,  $\varepsilon_x$  and  $\varepsilon_z$  (otherwise argument will be similar). This implies  $|x - z| \leq 2\varepsilon_x/3$  which means that  $(x - \varepsilon_x, x + \varepsilon_x)$  cannot be disjoint of  $K_2$ , which is a contradiction.

Because of Proposition 7 the value  $m(K)$ , with  $K \subset [-N, N]$  compact, can be written as

$$m(K) = \inf\{m(U) : U \subset [-N, N] \text{ open, and } K \subset U\}.$$

For a given  $\varepsilon > 0$  we therefore can choose open sets  $V_1, V_2, V_3$  in  $[-N, N]$  so that  $K_1 \subset V_1$ ,  $K_2 \subset V_2$ , and  $K_1 \cup K_2 \subset V_1 \cup V_2$ , and so that  $m(K_1) > m(V_1) - \varepsilon/3$ ,  $m(K_2) > m(V_2) - \varepsilon/3$ , and  $m(K_1 \cup K_2) > m(V_3) - \varepsilon/3$ .

By making  $V_1$  and  $V_2$  smaller if necessary, we just have proven, we can assume that  $V_1$  and  $V_2$  are disjoint.

Now we deduce from Proposition 4 that

$$\begin{aligned} m(K_1) + m(K_2) &\leq m(V_1 \cap V_3) + m(V_2 \cap V_3) \\ &\text{[since } K_1 \subset V_1 \cap V_3 \text{ and } K_2 \subset V_2 \cap V_3 \text{ and Proposition 5(d)]} \\ &= m((V_1 \cap V_3) \cup (V_2 \cap V_3)) \\ &\text{[By Proposition 4 (b)]} \\ &\leq m(V_3) \\ &\text{[By Proposition 4 (a)]} \\ &\leq \varepsilon/3 + m(K_1 \cup K_2) \\ &\text{[By Proposition 5 (d)]} \\ &\leq \varepsilon/3 + m(V_1 \cup V_2) \\ &= \varepsilon/3 + m(V_1) + m(V_2) \leq \varepsilon + m(K_1) + m(K_2) \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary it follows that  $m(K_1) + m(K_2) = m(K_1 \cup K_2)$ .  $\square$

From Lemma 10 we can easily deduce the following Corollary which is left as an exercise.

**Corollary 11.** *If  $U$  is open in  $[-N, N]$  and  $K$  is a compact subset of  $U$  then it follows that  $m(U \setminus K) = m(U) - m(K)$ .*

**Exercise 14.** *Prove Corollary 11.*

**Hint:** *Compute the measure of  $[-N, N] \setminus [U \setminus K]$  in two ways.*

*Proof of Theorem 9.* The claim (a) follows from the fact that  $\emptyset$  is open, and open sets are measurable by Proposition 7. The claim (b) was shown in Exercise 13 (a). We are left to show part (c). Therefore let  $A_n$  be measurable in  $[-N, N]$  for all  $n \in \mathbb{N}$ .

We define  $A = \bigcup_{n \in \mathbb{N}} A_n$ , and let  $\varepsilon > 0$ . We need to find a compact set  $K$  and open set  $U$  so that  $K \subset A \subset U$  and so that  $m(U) - m(K) < \varepsilon$ .

Using the definition of measurability for each set  $A_n$  with  $\varepsilon \cdot 2^{-n-1}$  instead of  $\varepsilon$  we can find compact sets  $K_n \subset A_n$  and open sets  $U_n$  with

$$m(U_n \setminus K_n) = m(U_n) - m(K_n) < \varepsilon \cdot 2^{-n-1}$$

( $m(U_n \setminus K_n) = m(U_n) - m(K_n)$  follows from Corollary 11).

We put  $U = \bigcup_{n=1}^{\infty} U_n$ , which is open and contains  $A$ . Since  $U$  is measurable there must be a compact set  $\tilde{K} \subset U$  so that  $m(U) - m(\tilde{K}) = m(U \setminus \tilde{K}) < \varepsilon/2$ .

Now  $\tilde{K} \subset U = \bigcup_{n \in \mathbb{N}} U_n$ , therefore, using compactness, we find an  $n_0 \in \mathbb{N}$  so that  $\tilde{K} \subset \bigcup_{n=1}^{n_0} U_n$ .

Finally put  $K = \bigcup_{n=1}^{n_0} K_n$  (the  $K_n$  were chosen above). The  $K$  is a compact set which is contained in  $A$ . We observe that

$$\begin{aligned} U \setminus K &\subset [U \setminus \bigcup_{n=1}^{n_0} U_n] \cup [\bigcup_{n=1}^{n_0} U_n \setminus K] \\ &= [U \setminus \bigcup_{n=1}^{n_0} U_n] \cup [\bigcup_{n=1}^{n_0} U_n \setminus \bigcup_{n=1}^{n_0} K_n] \subset [U \setminus \tilde{K}] \cup \bigcup_{n=1}^{n_0} (U_n \setminus K_n). \end{aligned}$$

Since  $m$  is subadditive on open sets (Proposition 4 part (c)) we deduce that

$$m(U) - m(K) \leq m(U \setminus \tilde{K}) + \sum_{i=1}^{n_0} m(U_n \setminus K_n) < \varepsilon/2 + \sum_{i=1}^{n_0} \varepsilon 2^{-n-1} < \varepsilon,$$

which finishes the proof of part (c) of Theorem 9.  $\square$

**Theorem 12.** *The map  $m : \mathcal{L}_{[-N, N]} \rightarrow [0, \infty)$ , defined in Equation (5) in has the following properties:*

- a)  $m$  is non-negative,
- b)  $m(\emptyset) = 0$ ,
- c)  $m$  is  $\sigma$ -additive, this means that: If  $A_n \in \mathcal{L}_{[-N, N]}$ , for  $n \in \mathbb{N}$ , and if the  $A_n$ 's are pairwise disjoint then

$$m\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} m(A_n).$$

*Proof.* (a) and (b) are clearly satisfied. In order to show (c) let  $A_n \in \mathcal{L}_{[-N, N]}$ ,  $n \in \mathbb{N}$ , be pairwise disjoint. Let  $A = \bigcup_{n \in \mathbb{N}} A_n$ . By (c) of Theorem 9 we already now that  $A$  is measurable.

For an arbitrary  $\varepsilon > 0$  we need to find a compact set  $K \subset A$  and a  $U \supset A$  which is open in  $[-N, N]$  so that

$$m(K) \geq -\varepsilon + \sum_{n \in \mathbb{N}} m(A_n) \text{ and } m(U) \leq \varepsilon + \sum_{n \in \mathbb{N}} m(A_n),$$

the first inequality then shows ( $\varepsilon > \text{arbitrary}$ ) that

$$m(A) = \sup_{n \in \mathbb{N}} \{m(\tilde{K}) : \tilde{K} \subset A \text{ compact}\} \geq \sum_{n \in \mathbb{N}} m(A_n),$$

while the second proves that

$$m(A) = \inf_{n \in \mathbb{N}} \{m(\tilde{U}) : \tilde{U} \supset A \text{ open}\} \leq \sum_{n \in \mathbb{N}} m(A_n).$$

We use the definition of measurability for each  $A_n$  and find compact sets  $K_n \subset A_n$  and in  $[-N, N]$  open sets  $U_n \supset A_n$  so that

$$m(U_n \setminus K_n) = m(U_n) - m(K_n) < \varepsilon \cdot 2^{-n-1}.$$

By definition of  $m(A_n)$ ,  $m(K_n) \leq m(A_n) \leq m(U_n)$ , and therefore

$$m(A_n) - m(K_n) < \varepsilon \cdot 2^{-n-1} \text{ and } m(U_n) - m(A_n) < \varepsilon \cdot 2^{-n-1}.$$

Since the  $K_n$ 's must be pairwise disjoint we can use Lemma 10 to observe that

$$\sum_{i \in \mathbb{N}} m(K_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n m(K_i) = \lim_{n \rightarrow \infty} m\left(\bigcup_{i=1}^n K_i\right) \leq 2N.$$

Thus, we can choose an  $n_0 \in \mathbb{N}$  so that

$$\begin{aligned} \sum_{i=1}^{n_0} m(K_i) &\geq -\varepsilon/2 + \sum_{i \in \mathbb{N}} m(K_i) \\ &\geq -\varepsilon/2 + \sum_{i \in \mathbb{N}} (m(A_i) - \varepsilon \cdot 2^{-i-1}) = -\varepsilon + \sum_{i \in \mathbb{N}} m(A_i). \end{aligned}$$

Therefore  $K = \bigcup_{i=1}^{n_0} K_i$  is a compact set contained in  $A$  with the wanted properties.

Secondly, choose  $U = \bigcup_{n \in \mathbb{N}} U_n$ . By Proposition 4 (note that  $V_n = \bigcup_{i=1}^n U_i$  increases to  $U$ )

$$\begin{aligned} m(U) &= \lim_{n \rightarrow \infty} m\left(\bigcup_{i=1}^n U_i\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n m(U_i) \\ &< \lim_{n \rightarrow \infty} \sum_{i=1}^n [m(A_i) + \varepsilon 2^{-1-i}] \leq \varepsilon + \sum_{i=1}^{\infty} m(A_i), \end{aligned}$$

which proves that  $U$  is an open set containing  $A$  with the wanted properties.  $\square$

We are now arriving at our last extension of our measure.

**Definition 13.** A set  $A \subset \mathbb{R}$  is called measurable if for all  $N \in \mathbb{N}$  the set  $A \cap [-N, N]$  is measurable in  $[-N, N]$ .

We denote the set of all measurable subsets of  $\mathbb{R}$  by  $\mathcal{L}$ , they are also called Lebesgue sets.

The Lebesgue measure is now defined

$$(6) \quad m : \mathcal{L} \rightarrow [0, \infty], \quad m(A) = \lim_{N \rightarrow \infty} m(A \cap [-N, N]).$$

Note that for  $N < N'$  it follows that  $m(A \cap [-N, N]) \leq m(A \cap [-N', N'])$  (since  $A \cap [-N', N']$  is the disjoint union of  $A \cap [-N, N]$  and  $A \cap ([-N', -N] \cup (N, N'])$ ). Therefore above limit exists in  $[0, \infty]$  and the value  $\infty$  could be achieved.

**Theorem 14.** The set  $\mathcal{L}$  of subsets of  $\mathbb{R}$  has the following properties:

- a)  $\emptyset \in \mathcal{L}$
- b)  $\mathcal{L}$  is closed under taking complements:

$$A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}.$$

- c)  $\mathcal{L}$  is closed under taking countable union: If  $A_n \in \mathcal{L}$ , for  $n \in \mathbb{N}$ , then  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{L}$ .

The map  $m : \mathcal{L} \rightarrow [0, \infty)$ , defined in (6) has the following properties:

- a)  $m$  is non-negative,
- b)  $m(\emptyset) = 0$ ,
- c)  $m$  is  $\sigma$ -additive, this means that: If  $A_n \in \mathcal{L}$ , for  $n \in \mathbb{N}$ , and if the  $A_n$ 's are pairwise disjoint then

$$m\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} m(A_n).$$

The proof follows easily from the corresponding properties of  $\mathcal{L}_{[-N, N]}$  and the properties of  $m$  defined on  $\mathcal{L}_{[-N, N]}$ .

**Exercise 15.** Prove Theorem 14.

**Exercise 16.** Show that the following sets are measurable and compute their measure.

$$\begin{aligned} &\mathbb{Q}, \\ &\mathbb{R} \setminus \mathbb{Q}, \\ &\bigcup_{n=1}^{\infty} [n, n + 2^{-n}] \\ &\bigcup_{n=1}^{\infty} [n, n + \frac{1}{n}] \end{aligned}$$

**Definition 15.** A subset  $A \subset \mathbb{R}$  is called null set if there is a set  $B \in \mathcal{L}$ , so that  $A \subset B$  and  $m(B) = 0$ .

**Exercise 17.** Show that every nullset is measurable (i.e. element of  $\mathcal{L}$ ).

**Exercise 18.** ( $m$  is translation invariant).

Let  $x \in \mathbb{R}$  and  $A \in \mathcal{L}$ .

Then  $x + A = \{x + a : a \in A\}$  is also in  $\mathcal{L}$  and  $m(x + A) = m(A)$ .

3. GENERAL MEASURES ON  $\sigma$ -ALGEBRAS

The following definition generalizes measurable sets on the real line and the Lebesgue measure defined for them.

**Definition 16.** Let  $X$  be a non-empty set.

A set of subsets  $\mathcal{M}$  of  $X$ , i.e.

$$\mathcal{M} \subset \mathcal{P}(X) = \{A : A \subset X\},$$

is called a  $\sigma$ -algebra on  $X$  if the following three conditions are satisfied:

- a)  $\emptyset \in \mathcal{M}$ ,
- b)  $A \in \mathcal{M} \Rightarrow A^c = X \setminus A \in \mathcal{M}$ ,
- c) If  $A_n \in \mathcal{M}$  for all  $n \in \mathbb{N} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$ .

In the case that  $\mathcal{M}$  is a  $\sigma$ -algebra on  $X$  we call  $(X, \mathcal{M})$  a measurable space.

For example  $(\mathbb{R}, \mathcal{L})$  is a measurable space.

If  $\mathcal{M}$  is a  $\sigma$ -algebra we call a map

$$\mu : \mathcal{M} \rightarrow [0, \infty] \quad (\text{the value } \infty \text{ is allowed})$$

a measure if

- a)  $\mu(\emptyset) = 0$ ,
- b) If  $A_n \in \mathcal{M}$  for all  $n \in \mathbb{N}$  and if the  $A_n$ 's are pairwise disjoint, then

$$\mu \left( \bigcup_{n \in \mathbb{N}} A_n \right) = \sum_{n \in \mathbb{N}} \mu(A_n).$$

**Exercise 19.** Let  $\mathcal{M}$  be a  $\sigma$ -algebra on a set  $X$ . Show that

- a)  $X \in \mathcal{M}$ .
- b)  $\mathcal{M}$  is closed under taking countable intersections.
- c)  $A, B \in \mathcal{M} \Rightarrow A \setminus B \in \mathcal{M}$ .
- d)  $A, B \in \mathcal{M} \Rightarrow A \Delta B = (A \setminus B) \cup (B \setminus A) \in \mathcal{M}$ .  
( $A \Delta B$  is called symmetric difference).

**Lemma 17.** (Continuity from below). For any measure  $\nu$  on  $(X, \mathcal{M})$  and an increasing sequence of sets  $(A_n)_{n=1}^{\infty}$  it follows that

$$\lim_{n \rightarrow \infty} \nu(A_n) = \nu \left( \bigcup_{n=1}^{\infty} A_n \right).$$

*Proof.* Write  $B_1 = A_1$ , and for  $n \geq 2$  define  $B_n = A_n \setminus A_{n-1}$ . Then  $(B_n)$  is a pairwise disjoint sequence of measurable sets, and it follows that  $A_n = \bigcup_{i=1}^n B_i$  and  $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ . Therefore:

$$\mu \left( \bigcup_{n=1}^{\infty} A_n \right) = \mu \left( \bigcup_{n=1}^{\infty} B_n \right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \mu(B_n) = \lim_{N \rightarrow \infty} \mu(A_N).$$

□

**Exercise 20.** Let  $\mathcal{M}$  be a  $\sigma$ -algebra on a set  $X$  and let  $\mu : \mathcal{M} \rightarrow [0, \infty]$  be a measure.

- a) For  $A, B \in \mathcal{M}$  we have  $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ .  
 b) (Continuity from above) If  $A_n$  is sequence in  $\mathcal{M}$  which decreases to  $\emptyset$ , and  $\mu(A_n) < \infty$ , for  $n \in \mathbb{N}$ , then

$$\lim_{n \rightarrow \infty} \mu(A_n) = 0.$$

**Exercise 21.** We want to show that in (c) of Exercise 20 the assumption “ $\mu(A_n) < \infty$ , for  $n \in \mathbb{N}$ ” is necessary.

Show that there is a sequence  $(A_n)$  of Lebesgue measurable subsets of  $\mathbb{R}$ , with  $m(A_n) = \infty$  for all  $n \in \mathbb{N}$ , so that  $A_n$  is decreasing to  $\emptyset$ .

**Exercise 22.** (The counting measure) Let  $X$  be a set and let  $\mathcal{M} = \mathcal{P}(X)$ , the set of all subsets (note, without writing down the proof,  $\mathcal{P}(X)$  is a  $\sigma$ -algebra).

For  $A \subset X$  define

$$\mu(A) = \begin{cases} \text{number of elements of } A & \text{if } A \text{ finite} \\ \infty & \text{if } A \text{ infinite} \end{cases}$$

Show that  $\mu$  is a measure.

**Exercise 23.** (The restriction of a measure) Assume  $(X, \mathcal{M})$  is a measurable space and  $\mu$  is a measure on  $\mathcal{M}$ . Fix  $A \in \mathcal{M}$ . Define

$$\mu|_A : \mathcal{M} \rightarrow [0, \infty], \quad B \mapsto \mu(A \cap B).$$

Show that also  $\mu|_A$  is measure on  $\mathcal{M}$ .

#### 4. MEASURABLE FUNCTIONS

**Definition 18.** Let  $(X, \mathcal{M})$  be a measurable space and let  $f : X \rightarrow [-\infty, \infty]$  be a function (we allow the values  $-\infty$  and  $\infty$ ).

We say that  $f$  is  $\mathcal{M}$ -measurable if

$$\forall a, b \in \mathbb{R}, a < b, \quad f^{-1}((a, b)) \in \mathcal{M}.$$

In the case  $X = \mathbb{R}$  and  $\mathcal{M} = \mathcal{L}$  we mean by  $f : \mathbb{R} \rightarrow \mathbb{R}$  is measurable that  $f$  is  $\mathcal{L}$ -measurable.

**Proposition 19.** Let  $(X, \mathcal{M})$  be a measurable space and let  $f : X \rightarrow \mathbb{R}$  be a function. The following are equivalent.

- a)  $f$  is  $\mathcal{M}$ -measurable,  
 b)  $\forall U \subset \mathbb{R}$  open  $f^{-1}(U) \in \mathcal{M}$ .  
 c)  $\forall F \subset \mathbb{R}$  closed  $f^{-1}(F) \in \mathcal{M}$ .  
 d)  $\forall a < b$  in  $\mathbb{R}$ ,  $f^{-1}([a, b]) \in \mathcal{M}$ .  
 e)  $\forall a \in \mathbb{R}$   $f^{-1}((a, \infty)) \in \mathcal{M}$ .  
 f)  $\forall b \in \mathbb{R}$   $f^{-1}((-\infty, b]) \in \mathcal{M}$ .

*Proof.* We will show that (a)  $\iff$  (b)  $\iff$  (e) and leave the other equivalences to the reader. (a)  $\Rightarrow$  (b) Let  $U \subset \mathbb{R}$  be open. By Theorem 3 we can write  $U$  as a countable union of disjoint intervals,  $U = \bigcup_{i=1}^{\infty} (a_i, b_i)$  (actually here we only need the representation of  $U$  as a countable union of open intervals, but not of disjoint ones). By assumption  $f^{-1}(a_i, b_i) \in \mathcal{M}$  and thus

$$f^{-1}(U) = \bigcup_{i=1}^{\infty} f^{-1}(a_i, b_i) \in \mathcal{M}$$

(b)  $\Rightarrow$  (e) clear since  $(a, \infty)$  open.

(e)  $\Rightarrow$  (a) Assume  $a < b$ . then

$$(a, b) = (-\infty, b) \setminus (-\infty, a] = (-\infty, b) \setminus \bigcap_{q \in \mathbb{Q}, q \geq a} (-\infty, q),$$

and thus

$$f^{-1}(a, b) = f^{-1}(-\infty, b) \setminus \bigcap_{q \in \mathbb{Q}, q \geq a} f^{-1}(-\infty, q) \in \mathcal{M}.$$

□

**Exercise 24.** Prove the other equivalences.

**Proposition 20.** If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous then  $f$  is measurable.

**Exercise 25.** Show that every piecewise continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is measurable.

**Proposition 21.** Let  $(X, \mathcal{M})$  is a measurable space. For  $A \subset X$ , define

$$1_A : X \rightarrow \mathbb{R}, \quad x \mapsto \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Then  $1_A$  is  $\mathcal{M}$ -measurable if and only if  $A \in \mathcal{M}$ .

*Proof.*  $\Rightarrow$ : If  $1_A$  is  $\mathcal{M}$ -measurable then in particular  $1_A^{-1}(1/2, 3/2) \in \mathcal{M}$ . But  $1_A^{-1}(1/2, 3/2) = A$ .

$\Leftarrow$  Assume  $A \in \mathcal{M}$ , and let  $a < b$ , then

$$1_A^{-1}(a, b) = \begin{cases} \emptyset & \text{if } 0, 1 \notin (a, b) \\ A & \text{if } 0 \leq a < 1 < b \\ A^c & \text{if } a < 0 < b \leq 1 \\ X & \text{if } a < 0 < 1 < b \end{cases}$$

Since  $\emptyset, X = \emptyset^c, A$  and  $A^c$  are in  $\mathcal{M}$  it follows that  $1_A$  is  $\mathcal{M}$ -measurable. □

**Proposition 22.** Let  $(X, \mathcal{M})$  be a measurable space the set of all  $\mathcal{M}$ -measurable maps on  $X$ , we denote it by,

$$L_0(X, \mathcal{M}) = \{f : X \rightarrow \mathbb{R} : f \text{ is } \mathcal{M}\text{-measurable}\},$$

is a vector space.

Moreover, if  $f$  and  $g$  are measurable maps on  $X$ , then

- a)  $f \cdot g$  is measurable,  
 b)  $\max(f, g)$ , and  $\min(f, g)$  are measurable.

*Proof.* Let  $f$  and  $g$  be  $\mathcal{M}$ -measurable functions on  $X$  to show that  $f + g$  is  $\mathcal{M}$ -measurable. Let  $a \in \mathbb{R}$ . By Proposition 19 we need to show that

$$(f + g)^{-1}(a, \infty) = \{x \in X : f(x) + g(x) > a\} \in \mathcal{M}.$$

$$\begin{aligned} (f + g)^{-1}(a, \infty) &= \{x \in X : f(x) + g(x) > a\} \in \mathcal{M} \\ &= \bigcup_{q \in \mathbb{Q}} \{x \in X : f(x) > q \text{ and } g(x) > a - q\} \\ &\supset: \text{ if for some } q \in \mathbb{Q} : f(x) > q \text{ and } g(x) > a - q \\ &\quad \text{then } f(x) + g(x) > a \\ &\subset: \text{ assume } f(x) + g(x) > a \\ &\quad \text{then } \exists \varepsilon > 0 \quad f(x) + g(x) > a + \varepsilon > a. \\ &\quad \text{Pick } q \in \mathbb{Q} \text{ so that } f(x) > q > f(x) - \varepsilon \\ &\quad \text{then } g(x) > a + \varepsilon - f(x) > a + \varepsilon - (q + \varepsilon) > a - q. \\ &= \bigcup_{q \in \mathbb{Q}} \{x \in X : f(x) > q\} \cap \{x \in X : g(x) > a - q\} \end{aligned}$$

Now  $\{x \in X : f(x) > q\}$  and  $\{x \in X : g(x) > a - q\}$  are elements of  $\mathcal{M}$ , therefore  $\{x \in X : f(x) > q\} \cap \{x \in X : g(x) > a - q\} \in \mathcal{M}$  for each  $q$  (since  $\sigma$ -algebras are closed under countable intersections). Since  $\mathbb{Q}$  is countable, and, since  $\sigma$ -algebras are closed under countable unions it follows that

$$\bigcup_{q \in \mathbb{Q}} \{x \in X : f(x) > q\} \cap \{x \in X : g(x) > a - q\} \in \mathcal{M}.$$

Secondly, we need to show that if  $f$  is a  $\mathcal{M}$ -measurable function on  $X$  and if  $c \in \mathbb{R}$ , then  $cf$  is measurable. We can assume that  $c \neq 0$  (the null-function is measurable).

For  $a \in \mathbb{R}$  note that

$$[cf]^{-1}(a, \infty) = \begin{cases} \{x \in X : f(x) > a/c\} & \text{if } c > 0 \\ \{x \in X : f(x) < a/c\} & \text{if } c < 0, \end{cases}$$

which implies that in both cases  $[cf]^{-1}(a, \infty)$  is in  $\mathcal{M}$ , and thus  $cf$  is measurable.

The fact, that  $f \cdot g$  is measurable whenever  $f$  and  $g$  are  $\mathcal{M}$ -measurable maps on  $X$ , is left as an exercise.  $\square$

**Exercise 26.** Show that that  $f \cdot g$ ,  $\min(f, g)$  and  $\max(f, g)$  are  $\mathcal{M}$ -measurable whenever  $f$  and  $g$  are  $\mathcal{M}$ -measurable maps on  $X$ .

**Proposition 23.** Let  $(f_n)$  be a sequence of  $\mathcal{M}$ -measurable maps on  $X$ .

Define for  $x \in X$

$$g(x) = \liminf_{n \rightarrow \infty} f_n(x) \text{ and } h(x) = \limsup_{n \rightarrow \infty} f_n(x).$$

(note that  $\liminf_{n \rightarrow \infty} f_n(x)$  and  $\limsup_{n \rightarrow \infty} f_n(x)$  always exists if we allow the values  $-\infty$  and  $\infty$ ).

Then  $g$  and  $h$  are measurable.

*Proof.* We will prove the measurability of  $g$  and leave the measurability of  $h$  as an exercise.

Let  $a \in \mathbb{R}$ , for  $x \in X$  it follows:

$$\begin{aligned} x \in g^{-1}(a, \infty) &\iff \liminf_{n \rightarrow \infty} f_n(x) > a \\ &\iff \sup_{n \in \mathbb{N}} \inf_{k \geq n} f_k(x) > a \\ &\iff \exists n \in \mathbb{N} \inf_{k \geq n} f_k(x) > a \\ &\iff \exists n \in \mathbb{N} \forall k \geq n \quad f_k(x) > a \\ &\iff x \in \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} f_k^{-1}(a, \infty), \end{aligned}$$

which means that the set  $g^{-1}(a, \infty)$  is a countable union of countable intersections of elements of  $\mathcal{M}$ , and, thus, in  $\mathcal{M}$ . The second part of the claim is proven in a similar way and is left as an exercise.  $\square$

**Exercise 27.** Let  $(f_n)$  be a sequence of  $\mathcal{M}$ -measurable maps on  $X$ . Prove that  $h = \limsup_{n \rightarrow \infty} f_n$  is measurable.

## 5. INTEGRAL OF MEASURABLE FUNCTION WITH RESPECT TO A MEASURE

Assume that  $(X, \mathcal{M})$  is a measurable space and  $\mu$  is a measure on  $\mathcal{M}$ . For a  $\mathcal{M}$ -measurable map  $f : X \rightarrow [-\infty, \infty]$  we want to define (if possible) the *integral of  $f$  with respect to  $\mu$*  and denote it by

$$\int f(x) d\mu(x).$$

In the case that  $\mathcal{M}$  is the  $\sigma$ -algebra of Lebesgue measurable subsets of  $\mathbb{R}$ ,  $\mu$  is the measure  $M$ , and  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable, we will show that the *trivial extension*

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in [a, b] \\ 0 & \text{if } x \notin [a, b]. \end{cases}$$

is measurable and  $\int \tilde{f}(x) dm(x)$  coincides with the Riemann integral  $\int_a^b f(x) dx$ .

**Definition 24.** A function  $f : X \rightarrow \mathbb{R}$  is called *simple function* if it is of the form

$$f(x) = \sum_{i=1}^n a_i 1_{A_i}(x),$$

where  $A_1, A_2, \dots, A_n$  are elements of  $\mathcal{M}$  and  $a_1, a_2, \dots, a_n \in \mathbb{R}$ .

**Proposition 25.** A function  $f : X \rightarrow [-\infty, \infty]$  is simple if and only if

- (1)  $f$  is  $\mathcal{M}$ -measurable,

(2) the range of  $f$  has only finitely many values.

Let  $z_1, z_2, \dots, z_n$  be the values of  $f$ . For  $i = 1, 2, \dots, n$  we define  $A_i = f^{-1}(\{z_i\})$ . Note that the  $A_i$ 's are pairwise disjoint, that  $\bigcup_{i=1}^n A_i = X$ , and that we can write  $f$  as

$$f = \sum_{i=1}^n z_i 1_{A_i}.$$

We call this the standard representation of  $f$ .

It is easy now to define the *integral of a non-negative simple function with respect to the measure  $\mu$* :

Assume that  $f : X \rightarrow [0, \infty]$  is simple and that  $z_1, z_2, \dots, z_n \in [0, \infty]$  are the possible values of  $f$ .

We let  $A_i = f^{-1}(\{z_i\})$  for  $i = 1, 2, \dots, n$ . By the proof of Proposition 25 we can write  $f$  as

$$f = \sum_{i=1}^n z_i 1_{A_i},$$

where  $(A_i)_{i=1}^n$  is a partition of  $X$  into measurable sets.

And we define the *integral of  $f$  with respect to  $\mu$*  the (possible infinite value)

$$(7) \quad \int f(x) d\mu(x) = \sum_{i=1}^n z_i \mu(A_i),$$

with the convention that we put  $z_i \mu(A_i) = 0$  if  $z_i = 0$  and  $\mu(A_i) = \infty$ .

**Example 26.** Let  $X = \mathbb{R}$ , and  $\mathcal{M}$  be the  $\sigma$ -algebra of Lebesgue sets and let  $\mu = m$ . If

$$f = \sum_{i=1}^n a_i 1_{I_i},$$

where the  $I_i$ 's are disjoint intervals, then

$$\int f(x) dm(x) = \sum a_i m(I_i) = \text{Riemann} - \int f(x) dx.$$

**Proposition 27.** Let  $f$  and  $g$  be simple non-negative functions on  $X$ .

- a) If  $c \geq 0$ ,  $\int cf(x) d\mu(x) = c \int f(x) d\mu(x)$ .
- b)  $\int f(x) + g(x) d\mu(x) = \int f(x) d\mu(x) + \int g(x) d\mu(x)$ .
- c) If  $f \leq g$  then  $\int f(x) d\mu(x) \leq \int g(x) d\mu(x)$ .
- d) The map

$$\mathcal{M} \rightarrow [0, \infty], \quad A \rightarrow \int 1_A f(x) d\mu(x),$$

is a measure.

*Proof.* Let

$$f = \sum_{i=1}^m a_i 1_{A_i} \text{ and } g = \sum_{i=1}^n b_i 1_{B_i},$$

be the standard representation of  $f$  and  $g$ .

(a) For  $c \geq 0$  we note that

$$cf = \sum_{i=1}^m ca_i 1_{A_i},$$

is the standard representation of  $cf$ , and therefore

$$\int cf(x) d\mu(x) = \sum_{i=1}^m ca_i \mu(A_i) = c \sum_{i=1}^m a_i \mu(A_i) = c \int f(x) d\mu(x).$$

(b) Since the  $A_i$ 's as well as the  $B_j$ 's are partitions of  $X$  we deduce that

$$A_i = \bigcup_{j=1}^n (A_i \cap B_j) \text{ for } i = 1, \dots, m \text{ and}$$

$$B_j = \bigcup_{i=1}^m (B_j \cap A_i) \text{ for } j = 1, \dots, n.$$

Therefore we deduce from the additivity of measures that:

$$\begin{aligned} \int f(x) d\mu + \int g(x) d\mu &= \sum_{i=1}^m a_i \mu(A_i) + \sum_{j=1}^n b_j \mu(B_j) \\ &= \sum_{i=1}^m \sum_{j=1}^n a_i \mu(A_i \cap B_j) + \sum_{j=1}^n \sum_{i=1}^m b_j \mu(A_i \cap B_j) \\ &= \sum_{i=1}^m \sum_{j=1}^n (a_i + b_j) \mu(A_i \cap B_j) \\ &= \int (f(x) + g(x)) d\mu(x) \end{aligned}$$

(for the last equality, first cancel all  $A_i \cap B_j$ 's which are empty and note that for the other choices of  $i$  and  $j$  the value  $a_i + b_j$  is in the range of  $f + g$ , then combine all  $A_i \cap B_j$ 's for which the values  $a_i + b_j$  are the same, note that you arrived to the standard representation of  $f + g$ ).

(c) We write again  $f$  and  $g$  using the sets  $A_i \cap B_j$  :

$$f = \sum_{i=1}^m a_i 1_{A_i} = \sum_{i=1}^m \sum_{j=1}^n a_i 1_{A_i \cap B_j} \text{ and}$$

$$g = \sum_{j=1}^n b_j 1_{B_j} = \sum_{j=1}^n \sum_{i=1}^m b_j 1_{A_i \cap B_j} = \sum_{i=1}^m \sum_{j=1}^n b_j 1_{A_i \cap B_j}.$$

If we assume that  $f(x) \leq g(x)$  for all  $x \in \mathbb{R}$  and note that the family  $(A_i \cap B_j)_{i=1, j=1}^{m, n}$  is a partition of  $X$  we deduce that for all choices of  $i \in \{1, \dots, m\}$  and all  $j \in \{1, \dots, n\}$  for which  $B_j \cap A_i$  is not empty it follows that  $a_i \leq b_j$ . Therefore

$$\int f(x) d\mu(x) = \sum_{i=1}^m \sum_{j=1}^n a_i \mu(A_i \cap B_j) \leq \sum_{i=1}^m \sum_{j=1}^n b_j \mu(A_i \cap B_j) = \int g(x) d\mu(x).$$

(d) First fix an element  $A \in \mathcal{M}$  and consider the map:

$$\mu|_A : \mathcal{M} \ni B \mapsto \mu(B \cap A).$$

It is easy to note that  $\mu|_A$  is also a measure on  $\mathcal{M}$ : Indeed  $\mu|_A(\emptyset) = \mu(\emptyset \cap A) = 0$ , and if  $(B_n)$  is a pairwise disjoint sequence in  $\mathcal{M}$ . Then also  $(B_n \cap A)$  is a pairwise disjoint sequence, and we deduce that

$$\mu|_A\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A \cap B_i\right) = \sum_{i=1}^{\infty} \mu(A \cap B_i) = \sum_{i=1}^{\infty} \mu|_A(B_i).$$

Secondly we note that if  $f = \sum_{i=1}^m a_i 1_{A_i}$  and  $B \in \mathcal{M}$ , then

$$\int 1_A f(x) d\mu(x) = \sum_{i=1}^m a_i \mu|_{A_i}(B).$$

Therefore our claim follows from the fact that positive linear combinations of measures are still measures.  $\square$

Now we are ready to define the integral for any non-negative measurable function on  $X$ :

**Definition 28.** Let  $f : X \rightarrow [0, \infty]$  be non-negative and measurable. We put

$$(8) \quad \int f(x) d\mu(x) = \sup \left\{ \int \phi d\mu : \phi \text{ is simple and } 0 \leq \phi \leq f \right\}.$$

**Theorem 29.** (The Monotone Convergence Theorem for non negative functions) If  $(f_n)$  is an increasing sequence of non-negative measurable functions on  $X$  (i.e.  $f_n(x) \leq f_{n+1}(x)$  for all  $x \in X$  and all  $n \in \mathbb{N}$ ), and we let

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \sup_{n \rightarrow \infty} f_n(x).$$

Then

$$\int f(x) d\mu(x) = \lim_{n \rightarrow \infty} \int f_n(x) d\mu(x).$$

*Proof.* First note that the integral for non-negative measurable functions is also monotone (because if  $f \leq g$  the value  $\int f d\mu$  is the supremum over a smaller set of integrals).

If  $(f_n)$  is increasing and  $f = \lim_{n \rightarrow \infty} f_n = \sup_{n \rightarrow \infty} f_n$ , it follows from the monotonicity that  $(\int f_n d\mu)$  is an increasing sequence of numbers and that  $\int f_n d\mu \leq \int f d\mu$  and therefore

$$\lim_{n \rightarrow \infty} \int f_n d\mu \leq \int f d\mu.$$

To establish the reverse inequality, fix  $\alpha \in (0, 1)$ , let  $\phi$  be a simple functions with  $0 \leq \phi \leq f$ . Define for  $n \in \mathbb{N}$  the set:

$$E_n = \{x \in X : f_n(x) \geq \alpha\phi(x)\}.$$

Since for all  $x$  the  $\lim_{n \rightarrow \infty} f_n(x) = f(x) \geq \phi(x) \geq \alpha\phi(x)$  (the last inequality being strict whenever  $\phi(x) > 0$ ), we deduce that  $E_n$  is an increasing sequence in  $\mathcal{M}$  whose union is all of  $X$ .

From the monotonicity of  $\int$  it follows that

$$\int f_n d\mu \geq \int 1_{E_n} f_n d\mu \geq \alpha \int 1_{E_n} \phi d\mu.$$

Since by Proposition 27 (d), the map

$$\mathcal{M} \ni A \mapsto \int 1_A \phi d\mu$$

is a measure, it follows from the continuity from below (Lemma 17) that

$$\int \phi d\mu = \lim_{n \rightarrow \infty} \int 1_{E_n} \phi d\mu.$$

Therefore we deduce that

$$\alpha \int \phi d\mu = \alpha \lim_{n \rightarrow \infty} \int 1_{E_n} \phi d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Now we note that since  $\phi$  was an arbitrary simple function for which  $0 \leq \phi \leq f$ , it follows that

$$\alpha \int f d\mu = \alpha \sup_{\phi \text{ simple } 0 \leq \phi \leq f} \int \phi f d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Secondly, since  $\alpha < 1$  was arbitrary we deduce that (let  $\alpha \nearrow 1$ ) that

$$\int f d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu,$$

which finishes the proof of the theorem.  $\square$

**Proposition 30.** *If  $f : X \rightarrow [0, \infty]$  is measurable, then there exists an increasing sequence of simple non-negative functions  $(\phi_n)$ , so that for all  $x \in X$ :*

$$f(x) = \lim_{n \rightarrow \infty} \phi_n = \sup_{n \in \mathbb{N}} \phi_n(x),$$

and therefore by Theorem 29,

$$\int f(x) d\mu(x) = \lim_{n \rightarrow \infty} \int f_n d\mu(x) = \sup_{n \in \mathbb{N}} \int f_n(x) d\mu(x).$$

*Proof.* Let  $f : X \rightarrow [0, \infty]$  be measurable. For  $n \in \mathbb{N}$  we define the following function  $f_n$ :

$$f_n(x) = n1_{f^{-1}(n, \infty]} + \sum_{k=1}^{n2^n} (k-1) \cdot 2^{-n} 1_{f^{-1}((k-1) \cdot 2^{-n}, k \cdot 2^{-n}]}$$

Note that

$$\begin{aligned} f_{n+1}(x) &= (n+1)1_{f^{-1}(n+1, \infty]} + \sum_{k=1}^{(n+1)2^{n+1}} (k-1) \cdot 2^{-n-1} 1_{f^{-1}((k-1) \cdot 2^{-n-1}, k \cdot 2^{-n-1})} \\ &\geq n1_{f^{-1}(n, \infty]} + \sum_{k=1}^{n2^{n+1}} (k-1) \cdot 2^{-n-1} 1_{f^{-1}((k-1) \cdot 2^{-n-1}, k \cdot 2^{-n-1})} \\ &= n1_{f^{-1}(n, \infty]} + \sum_{\ell=1}^{n2^n} (2\ell-2) \cdot 2^{-n-1} 1_{f^{-1}((2\ell-2) \cdot 2^{-n-1}, (2\ell-1) \cdot 2^{-n-1})} \\ &\quad + \sum_{\ell=1}^{n2^n} ((2\ell-1) \cdot 2^{-n-1} 1_{f^{-1}((2\ell-1) \cdot 2^{-n-1}, 2\ell \cdot 2^{-n-1})} \\ &\geq n1_{f^{-1}(n, \infty]} + \sum_{\ell=1}^{n2^n} (\ell-1) \cdot 2^{-n} 1_{f^{-1}((\ell-1) \cdot 2^{-n}, \ell \cdot 2^{-n})} = f_n(x), \end{aligned}$$

which means that  $(f_n)$  is an increasing sequence. Secondly it follows for an  $x \in X$  for which  $f(x) \leq n$  that  $0 \leq f(x) - f_n(x) \leq 2^{-n}$ . Thus it follows that  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for all  $x \in X$ .  $\square$

**Corollary 31.** *Let  $f$  and  $g$  be measurable non-negative functions on  $X$  and let  $c \geq 0$ . Then it follows that*

- a)  $\int f(x) + g(x) d\mu(x) = \int f(x) d\mu(x) + \int g(x) d\mu(x)$ , and
- b)  $\int cf(x) d\mu(x) = c \int f(x) d\mu(x)$ .

**Definition 32.** *Let  $f$  and  $g$  be two measurable functions on  $X$  we say that  $f$  is  $\mu$ -almost everywhere equal to  $g$  if*

$$\{x \in X : f(x) \neq g(x)\} \in \mathcal{M} \text{ and } \mu(\{x \in X : f(x) \neq g(x)\}) = 0.$$

*In that case we write:*

$$f = g \quad \mu\text{-a.e.}$$

**Proposition 33.** *If  $f$  is a measurable and non-negative function then*

$$\int f d\mu = 0 \iff f = 0 \mu\text{-a.e.}$$

*Proof.* “ $\Leftarrow$ ” For every non-negative simple function  $\phi$ , say

$$\phi = \sum_{i=1}^n a_i 1_{A_i} \text{ with } (A_i) \text{ partition of } X,$$

with  $0 \leq \phi \leq f$  it follows that

$$\forall i \in \{1, \dots, n\} \text{ Either } a_i = 0 \text{ or } \mu(A_i) = 0.$$

Thus  $\int f d\mu = 0$ .

“ $\Rightarrow$ ”: For  $n \in \mathbb{N}$  define  $E_n = \{x \in X : f(x) > 1/n\}$ . Since

$$\frac{1}{n}1_{E_n} \leq f.$$

It follows that

$$0 \leq \int \frac{1}{n}1_{E_n} d\mu = \frac{1}{n}\mu(E_n) \leq \int f d\mu.$$

Thus  $\mu(E_n) = 0$ . Since  $(E_n)$  is an increasing sequence of sets whose union is  $\{x \in X : f(x) \neq 0\}$ . Finally it follows from Lemma 17, that:

$$\mu(\{x \in X : f(x) \text{ not } = 0\}) = \lim_{n \rightarrow \infty} \mu(E_n) = 0.$$

□

**Corollary 34.** *Assume that  $(f_n)$  is a  $\mu$ -a.e. increasing sequence of measurable functions, i.e.*

$$\mu(\{x \in X : \exists n \in \mathbb{N} f_n(x) > f_{n+1}(x)\}) = 0.$$

Then it follows for  $f(x) = \sup_n f_n(x)$ , that

$$\lim_{n \rightarrow \infty} \int f_n d\mu(x) = \int f d\mu(x).$$

**Theorem 35.** (Fatou's Lemma) *Let  $(f_n)$  be a sequence of non-negative measurable functions on  $X$ . Then*

$$\int \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

*Proof.* For each  $k \in \mathbb{N}$  and each  $j \geq k$  we have  $\inf_{i \geq k} f_i \leq f_j$ , which implies by the monotonicity of integrals that  $\int \inf_{i \geq k} f_i d\mu \leq \int f_j d\mu$  and thus

$$\int \inf_{i \geq k} f_i d\mu \leq \inf_{j \geq k} \int f_j d\mu.$$

Taking on both sides now the limit for  $k \rightarrow \infty$  implies our claim. □

**Definition 36.** (Integral for measurable functions)

Now let  $f : X \rightarrow \mathbb{R}$  be any measurable function. We define

$$f^+(x) = \max(f(x), 0) \text{ and } f^- = -\min(f(x), 0).$$

Note that  $f^+$  and  $f^-$  are both non-negative measurable functions on  $X$ , that  $f = f^+ - f^-$  and that  $|f| = f^+ + f^-$ .

We call  $f$   $\mu$ -integrable if

$$\int f^+ d\mu < \infty \text{ and } \int f^- d\mu < \infty \quad (\iff \int |f| d\mu < \infty),$$

and put in this case

$$(9) \quad \int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

We define

$$\mathcal{L}_1(\mu) = \left\{ f : X \rightarrow \mathbb{R}, f \text{ is } \mathcal{M}\text{-measurable and } \int |f| d\mu < \infty \right\}.$$

**Proposition 37.**  $\mathcal{L}_1(\mu)$  is a vector space and the integral is a linear and monotone map on  $\mathcal{L}_1(\mu)$ .

*Proof.* The claim that  $\mathcal{L}_1(\mu)$  is vector space follows from the fact that for  $f, g \in \mathcal{L}_1(\mu)$  and  $\alpha, \beta \in \mathbb{R}$ ,

$$|\alpha f(x) + \beta g(x)| \leq |\alpha| \cdot |f(x)| + |\beta| \cdot |g(x)|.$$

It is also easy to check that

$$\alpha \int f d\mu = \int \alpha f d\mu.$$

The problem with showing the additivity of  $\int$  is that we do not necessarily have  $(f+g)^+ = f^+ + g^+$  or  $(f+g)^- = f^- + g^-$ . So we argue as follows:

Let  $h = f+g$ . Then  $h^+ - h^- = h = f^+ - f^- + g^+ - g^- = f^+ + g^+ - f^- - g^-$ . Thus  $h^+ + f^- + g^- = h^- + f^+ + g^+$ , which implies by the additivity of the integral on non-negative functions that

$$\int h^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int h^- d\mu + \int f^+ d\mu + \int g^+ d\mu,$$

thus

$$\begin{aligned} \int h d\mu &= \int h^+ d\mu - \int h^- d\mu \\ &= \int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu + \int g^- d\mu \\ &= \int f d\mu + \int g d\mu. \end{aligned}$$

Thus we showed that  $\int \cdot d\mu$  is linear. In order to show monotonicity assume that

$$f = f^+ - f^- \leq g^+ - g^- = g.$$

The  $f^+ + g^- \leq g^+ + f^-$  and from the monotonicity and linearity of  $\int \cdot d\mu$  for non negative function it follows that

$$\int f^+ + g^- d\mu = \int f^+ d\mu + \int g^- d\mu \geq \int g^+ d\mu + \int f^- d\mu = \int g^+ + f^- d\mu,$$

which implies that

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu \leq \int g d\mu = \int g^+ d\mu - \int g^- d\mu$$

□

**Exercise 28.** Prove the Monoton Convergence Theorem for general functions, i.e. let  $(f_n) \subset \mathcal{L}_1(\mu)$  be a  $\mu$ -a.e. increasing sequence. And let

$$f(x) = \lim_{n \rightarrow \infty} f_n(x),$$

wherever this limit exists, then

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \lim_{n \rightarrow \infty} \int f d\mu.$$

**Theorem 38.** (The Dominated Convergence Theorem) Assume that  $(f_n)$  is a sequence in  $\mathcal{L}_1(\mu)$  such that

- a)  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  exists  $\mu$ -a.e.
- b) There is a  $g \in \mathcal{L}_1(\mu)$  so that  $|f_n| \leq |g|$   $\mu$ -a.e..

Then

$$\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

*Proof.* After redefining the  $f_n$ 's  $f$  and  $g$  on a null set, it follows that  $|f| \leq g$ , and thus that  $f \in L(\mu)$ .

Since  $g + f_n \geq 0$  and  $g - f_n \geq 0$   $\mu$ -a.e. it follows from Fatou's lemma (Theorem 35) that

$$\begin{aligned} \int g d\mu + \int f d\mu &\leq \liminf_{n \rightarrow \infty} \int g + f_n d\mu = \int g d\mu + \liminf_{n \rightarrow \infty} \int f_n d\mu, \\ \int g d\mu - \int f d\mu &\leq \liminf_{n \rightarrow \infty} \int g - f_n d\mu = \int g d\mu - \limsup_{n \rightarrow \infty} \int f_n d\mu. \end{aligned}$$

Thus,

$$\liminf_{n \rightarrow \infty} \int f_n \geq \int f d\mu \geq \limsup_{n \rightarrow \infty} \int f_n,$$

which implies the claim. □