CONSTRUCTION OF THE LEBESGUES MEASURE, AN OVERVIEW

Main Goal. Let \( n \in \mathbb{N} \) and let \( \mathcal{B}_{\mathbb{R}^n} \) be the Borel sets on \( \mathbb{R}^n \), i.e. the \( \sigma \)-Algebra generated by the open subsets of \( \mathbb{R}^n \).

We want to show that there is one, and only one, measure \( m \) on \( \mathcal{B}_{\mathbb{R}^n} \) which assigns to an \( n \)-dimensional box its volume, i.e.

\[
m\left( \prod_{i=1}^{n} [a_i, b_i] \right) = \prod_{i=1}^{n} (b_i - a_i), \text{ whenever } -\infty < a_i < b_i < \infty.\]

Definition 1. Elementary Systems
Let \( X \) be a set. A system \( \mathcal{E} \subset \mathcal{P}(X) \) is called an elementary system on \( X \) if

1. \( \emptyset \in \mathcal{E} \),
2. \( \mathcal{E} \) is stable under finite intersections, i.e.
   - if \( E, F \in \mathcal{E} \) then \( E \cap F \in \mathcal{E} \), and
3. if \( E \in \mathcal{E} \) then \( E^c \) is the finite union of disjoint elements of \( \mathcal{E} \).

Example 2. On \( \mathbb{R}^n \) the following are elementary systems which generate \( \mathcal{B}_{\mathbb{R}^n} \):

(a) \( \mathcal{E} = \left\{ \prod_{i=1}^{n} [a_i, b_i] : -\infty \leq a_i < b_i \leq \infty \text{ for } i = 1, 2 \ldots n \right\} \)

(Replace \( [a_1, b_1] \) by \( (a_1, b_1) \) if \( a_1 = -\infty \))

(b) \( \mathcal{E} = \left\{ \prod_{i=1}^{n} (a_i, b_i] : -\infty \leq a_i < b_i \leq \infty \text{ for } i = 1, 2 \ldots n \right\} \)

(Replace \( (a_n, b_n] \) by \( (a_n, b_n) \) if \( b_n = \infty \))

If \( (X, \mathcal{M}) \) and \( (Y, \mathcal{N}) \) are measurable spaces, then

\[ \mathcal{E} = \{ A \times B : A \in \mathcal{M} \text{ and } B \in \mathcal{N} \} \]

is an elementary system on \( X \times Y \) which generates the product \( \sigma \)-algebra.
Proposition 3. If $\mathcal{E}$ is an elementary system on $X$, then

$$\mathcal{A} = \left\{ \bigcup_{i=1}^{n} E_i : n \in \mathbb{N}_0 \text{ and } E_1, E_2, \ldots, E_n \in \mathcal{E} \text{ are pairwise disjoint} \right\}$$

is an algebra on $X$.

**Definition.** Outer Measures.

An outer measure on a set $X$ is a map $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

1. $\mu^*(\emptyset) = 0$,
2. (Monotonicity) if $A \subset B$, then $\mu^*(A) \subset \mu^*(B)$,
3. (Countable Subadditivity) $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

**Definition.** $\mu^*$-mesurable.

If $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure a set $A \subset X$ is called $\mu^*$-mesurable if

$$\text{for all } E \subset X: \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

(Note: $\leq$ is always true, because of subadditivity)

We put $\mathcal{M}_{\mu^*} = \{ A \subset X : A \text{ is } \mu^*$-measurable $\}$.

**Proposition 4.** Let $\mathcal{E} \subset \mathcal{P}(X)$, $\emptyset, X \in \mathcal{E}$ and $\rho : \mathcal{E} \to [0, \infty]$, with $\rho(\emptyset) = 0$. Then

$$\mu^* : \mathcal{P} \to [0, \infty], \text{ defined by}$$

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : E_i \in \mathcal{E}, \text{ for } i \in \mathbb{N}, \text{ and } A \subset \bigcup_{i=1}^{\infty} E_i \right\}, \text{ for } A \subset X,$$

is an outer measure.

**Theorem 5.** (Carathéodory’s Theorem)

Let $\mu^*$ be an outer measure, then $\mathcal{M}_{\mu^*}$ is a $\sigma$-algebra, and $\mu^*$ restricted to $\mathcal{M}_{\mu^*}$ is a complete measure on $\mathcal{M}_{\mu^*}$.

**Remark 6.** Let us start with some $\mathcal{E} \subset \mathcal{P}(X)$, containing $\emptyset$ and $X$, and let $\rho : \mathcal{E} \to [0, \infty]$, with $\rho(\emptyset) = 0$. Define the outer measure $\mu^*$ as in Proposition 4.

Then $\mu^*$ on $\mathcal{M}_{\mu^*}$ is a measure. But of course we want two more properties: that $\mathcal{M}_{\mu^*}$ contains $\mathcal{E}$ and that $\mu^*$ coincides with $\rho$ on $\mathcal{E}$.

**Definition.** Premeasures

Let $\mathcal{A}$ be an algebra and let $\mu_0 : \mathcal{A} \to [0,\infty]$. Then $\mu_0$ is called a premeasure on $\mathcal{A}$ if

1. $\mu_0(\emptyset) = 0$, 

(2) if \((A_j)_{j=1}^{\infty}\) is a sequence of disjoint sets in \(\mathcal{A}\) such that \(\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}\), then
\[
\mu_0\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu_0(A_j).
\]

**Remark 7.** Note, that in order to have any chance to extend a map \(\mu_0 : \mathcal{A} \rightarrow [0, \infty]\) to a measure on the \(\sigma\)-algebra generated by \(\mathcal{A}\), \(\mu_0\) needs to be a premeasure. That this condition is sufficient follows from the following theorem.

**Theorem 8.** Let \(\mu_0\) be a premeasure on \(\mathcal{A}\), which is an algebra on a set \(X\). As before define an outer measure by
\[
\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \rho(E_i) : E_i \in \mathcal{A}, \text{ for } i \in \mathbb{N} \text{ and } A \subset \bigcup_{i=1}^{\infty} E_i \right\}, \text{ for } A \subset X.
\]
Then \(\mathcal{A} \subset \mathcal{M}_{\mu^*}\), the \(\mu^*\)-measurable sets, and \(\mu^*|_A = \mu_0\). Thus \(\mu_0\) can be extended to a measure on a \(\sigma\)-algebra, which contains the \(\sigma\)-algebra generated by \(\mathcal{A}\).

Getting back to the special case \(X = \mathbb{R}^n\), we still need to show that “volumes” are premeasures. This will yield the following more general result.

**Theorem 9.** The case \(\mathbb{R}^n\).

Let \(F : \mathbb{R}^n \rightarrow \mathbb{R}\) have the following properties:

1. \(F\) is (not necessarily strictly) increasing, i.e. if \(\bar{x} = (x_1, x_2, \ldots, x_n)\) and \(\bar{y} = (y_1, y_2, \ldots, y_n)\) are in \(\mathbb{R}^n\), so that \(x_i \leq y_i\) for all \(i = 1, 2, \ldots, n\) (we write in that case \(\bar{x} \leq \bar{y}\)), then \(F(\bar{x}) \leq F(\bar{y})\).

2. \(F\) is left continuous, i.e. for \((a_1, a_2, \ldots, a_n) \in \mathbb{R}^n\)
\[
\lim_{\substack{x_1 \to a_1, \ldots, x_n \to a_n \\text{in} \mathbb{R}^n}} F(x_1, x_2, \ldots, x_n) = F(a_1, a_2, \ldots, a_n)
\]

If \((a_1, a_2, \ldots, a_n) \in \mathbb{R} \cup \{\pm \infty\}\), put for \(k \in \mathbb{N}\)
\[
x_i^{(k)} = \begin{cases} a_i & \text{if } i \notin \{i : a_i = \pm \infty\} \\ k & \text{if } i \in \{i : a_i = \infty\} \\ -k & \text{if } i \in \{i : a_i = -\infty\} \end{cases}
\]
and then put
\[
F(a_1, a_2, \ldots, a_n) = \lim_{k \to \infty} F(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}).
\]
Let  
\[ E = \{ \prod_{i=1}^{n} [a_i, b_i] : -\infty \leq a_i \leq b_i \leq \infty \text{ for } i = 1, 2, \ldots, n \} \], and

\[ A = \{ \bigcup_{i=1}^{n} E_i : n \in \mathbb{N}_0 \text{ and } E_1, E_2, \ldots, E_n \in E \text{ are pairwise disjoint} \}. \]

For \( E = \prod_{i=1}^{n} [a_i, b_i] \in E \) put

\[ \mu_0(E) = \prod_{i=1}^{n} (F(b_i) - F(a_i)) \]

and if \( A = \bigcup_{j=1}^{m} E_j : n \in \mathbb{N}_0 \in A \), with \( E_1, E_2, \ldots, E_m \in E \) pairwise disjoint, put

\[ \mu_0(A) = \sum_{j=1}^{m} \mu_0(E_j). \]

Then \( \mu_0 \) is well defined, i.e. writing \( A \in \mathcal{A} \) in different ways as union of disjoint elements of \( E \), will lead to the same number for \( \mu(A) \), and \( \mu_0 \) is a premeasure.

**Remark 10.** Even in the simplest case \( n = 1 \) and \( F(x) = x \) it is necessary to use the completeness of \( \mathbb{R} \) to prove Theorem 8. The proof quickly reduces to show (in dimension 1) that if the interval \( I = [a, b) \) is written as countable union of pairwise disjoint half open intervals \( I_j \), then the length of \( I \) equals to sum of the lengths of the \( I_j \)'s. As "obvious" as this may seem at first sight, it is not, and is wrong if we replace \( \mathbb{R} \) by \( \mathbb{Q} \) (see exercise 1 on third homework sheet).

Combining everything we have so far (letting \( F(x) = x \) in Theorem 9).

**Corollary 11.** There is a \( \sigma \) algebra \( \mathcal{M} \) on \( \mathbb{R}^n \), \( n \in \mathbb{N} \), which contains the Borel sets, and there is a complete measure \( \mu \) \( \mathcal{M} \) on \( \mathcal{M} \), so that

\[ \mu \left( \prod_{i=1}^{n} [a_i, b_i] : \right) = \prod_{i=1}^{n} (b_i - a_i) \]

whenever \( -\infty \leq a_i \leq b_i \leq \infty \) for \( i = 1, 2, \ldots, n \).

Our next, and final, step is to show that there is only one measure on \( \mathcal{B}_{\mathbb{R}^n} \) which satisfies (*)}. It will follow from the following general Theorem:

**Theorem 12.** Let \( E \subset \mathcal{P}(X) \) be closed under taking finite intersection and assume that there is a monotone sequence \( (E_n) \subset E \) so that \( \bigcup_{n=1}^{\infty} E_n = X \).
Let $\mathcal{M}$ be the $\sigma$-algebra generated by $\mathcal{E}$. If $\mu$ and $\nu$ are two measures on $\mathcal{M}$ which coincide on $\mathcal{E}$, and for which $\mu(E_n) = \nu(E_n) < \infty$, $n \in \mathbb{N}$, then they are equal.

**Corollary 13.** There is a unique measure $m$ on $\mathcal{B}_{\mathbb{R}^n}$, so that

$$
\mu\left(\prod_{i=1}^{n}[a_i, b_i)\right) = \prod_{i=1}^{n}(b_i - a_i)
$$

whenever $-\infty \leq a_i \leq b_i \leq \infty$ for $i = 1, 2, \ldots, n$.

By a theorem in homework/text we can extend this measure uniquely to a complete measure on the completion of $\mathcal{B}_{\mathbb{R}^n}$ with respect to $m$.

$$
\mathcal{L} := \{A \cup N : A \in \mathcal{B}_{\mathbb{R}^n} \text{ and } N \text{ $m$-nullset}\}.
$$

We still denote the completed measure by $m$. $\mathcal{L}$ is called the *Lebesgues $\sigma$-algebra* and $m$ is called the *Lebesgues measure* on $\mathbb{R}^n$.

**Remark 14.** In order to show Theorem 12 we will show that the set

$$
\tilde{\mathcal{M}} = \{A \in \mathcal{M} : \mu(A) = \nu(A)\}
$$

is a $\sigma$-algebra. Unfortunately this is not possible, at least not directly. Knowing that $A_n \in \tilde{\mathcal{M}}$, for $n \in \mathbb{N}$, doesn’t seem to yield that $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$, unless the $A_n$’s are disjoint.

Therefore we will have to take a detour:

**Definition.** Let $X$ be a set. A system $\mathcal{D} \subset \mathcal{P}(X)$ is called a *Dynkin system* if

1. $\emptyset, X \in \mathcal{D}$,

2. if $A, B \in \mathcal{D}$ and $A \subset B$ then $B \setminus A \in \mathcal{D}$,

3. if $A_n \in \mathcal{D}$, $n \in \mathbb{N}$, and if the $A_n$’s are pair wise disjoint, then $\bigcup A_n \in \mathcal{D}$.

As for $\sigma$-algebras the intersection of (any number of) Dynkin systems is again a Dynkin system. Thus for $\mathcal{E} \subset \mathcal{P}(X)$

$$
\mathcal{D}(\mathcal{E}) = \bigcap_{\mathcal{D} \subset \mathcal{D}, \mathcal{D} \text{ is Dynk.}} \mathcal{D},
$$

is Dynkin system, and we call it the *Dynkin system generated by $\mathcal{E}$*.

**Proposition 15.** A Dynkin system which is closed under taking finite intersections is a $\sigma$-algebra.

**Theorem 16.** If $\mathcal{E}$ is closed under taking finite intersections, then $\mathcal{D}(\mathcal{E})$ is also closed under finite intersections, and thus a $\sigma$-algebra.
Thus, in order to prove Theorem 12, we need to verify that

$$\mathcal{M} = \{ A \in \mathcal{M} : \mu(A) = \nu(A) \}$$

is a Dynkin system.