HOW TO PROOF SOMETHING FOR ALL $f \in L_1$

Goal: Assume we have a measure space $(X, \mathcal{M}, \mu)$ and we would like to show that all $f \in L_1(\mu)$ or/and all $f \in L_0^+$ (non negative $\mathcal{M}$ measurable functions on X) have a certain property (*).

As an example of such a result we would like to show the Theorems of Tonelli and Fubini.

We will need the following notation.

Definition 1. Let X and Y be sets and $f : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ for x and y we define *the x-cut and the y-cut of f* by

$$f_x : Y \to \mathbb{R} \cup \{\pm\infty\}, \quad y \mapsto f(x, y)$$
$$f_y : X \to \mathbb{R} \cup \{\pm\infty\}, \quad x \mapsto f(x, y).$$

Theorem 2. *Assume $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \mu)$ are two σ-finite measure spaces.*

a) *(Theorem of Tonelli) If $f : X \times Y \to [0, \infty]$ is $\mathcal{M} \otimes \mathcal{N}$- measurable. Then f_x is $\mathcal{N}$ measurable, for all $x \in X$, and f_y is $\mathcal{M}$ measurable, for all $y \in Y$. Secondly*

$$\phi : X \to \mathbb{R} \cup \{\infty\}, \quad x \mapsto \int f_x d\nu, \text{ is } \mathcal{M} \text{ measurable, and}$$
$$\psi : Y \to \mathbb{R} \cup \{\infty\}, \quad y \mapsto \int f_y d\mu, \text{ is } \mathcal{N} \text{ measurable}$$

Moreover

$$\int \phi d\mu = \int \psi d\nu = \int f(x, y) d(\mu \otimes \nu).$$

Note that

$$\int \phi d\mu = \int \Big[\int f(x, y) d\nu \Big] d\mu, \text{ and}$$
$$\int \psi d\mu = \int \Big[\int f(x, y) d\mu \Big] d\nu.$$

b) *(Theorem of Fubini) If $f \in L_1(\mu \otimes \nu)$, then $f_x \in L_1(\nu)$ for μ-almost all $x \in X$, and $f_y \in L_1(\mu)$ for ν-almost all $y \in Y$. Secondly*

$$\phi : X \to \mathbb{R}, \quad x \mapsto \begin{cases} \int f_x d\nu, & \text{if integral exists in } \mathbb{R}, \\ 0 & \text{otherwise} \end{cases}$$
$$\psi : Y \to \mathbb{R}, \quad y \mapsto \begin{cases} \int f_y d\mu, & \text{if integral exists in } \mathbb{R}, \\ 0 & \text{otherwise} \end{cases}$$

are in $L_1(\mu)$, respectively $L_1(\nu)$. Moreover

$$\int \phi d\mu = \int \psi d\nu = \int f(x,y) d(\mu \otimes \nu).$$

In order to show Theorem 2 we need the following general, and very useful observation.

Theorem 3. *(Monotone Class Theorem for L_0^+ and L_1)*
Let $(Z, \mathcal{R}, \rho)$ be a measure space and let $\mathcal{E}$ be a nonempty $\cap$-stable generator of $\mathcal{M}$ containing a sequence (E_n) with $E_n \nearrow X$.

a) *Let $F^+ \subset L_0^+$ have the following properties.*
   i) $\{\chi_E : E \in \mathcal{E}\} \subset F^+$.
   ii) *If $A, B \in \mathcal{R}$, $A \subset B$ and χ_A, $\chi_B \in F^+$, then $\chi_B - \chi_A \in F^+$.*
   iii) *If $f, g \in F^+$ and $a, b \geq 0$ then $af + bg \in F^+$.*
   iv) *If $(f_n) \subset F^+$ and $f_n \nearrow f$, then $f \in F^+$.*

   Then $F^+ = L_0^+$.

b) *Let $F \subset L_1(\rho)$ have the following properties*
   i) $L_1^+(\rho) := L_1 \cap L_0^+(\rho) \subset F$.
   ii) *$f, g \in F^+$, $f, g \geq 0$, then $f - g \in F$.*

   Then $F = L_1(\rho)$.

The proof of Theorem is a straight forward application of Dynkin's theorem and the fact that we can approximate non negative measurable functions by measurable step functions.

Proof. (a) Define

$$\mathcal{D} := \{A \in \mathcal{R} : \chi_D \in F^+\}.$$

By (i) $\mathcal{E} \subset \mathcal{D}$. Since $\mathcal{E}$ is non empty, by using $a = b = 0$ in (iii) we deduce that $\emptyset \in \mathcal{D}$. Since $E_n \in \mathcal{E}$, $n \in \mathbb{N}$, and $E_n \nearrow X$, we deduce from (iv) that $X \in \mathcal{D}$. By (iii) $\mathcal{D}$ is closed under finite disjoint unions, and, thus, using also (iv) $\mathcal{D}$ is closed under disjoint countable unions. We proved therefore that $\mathcal{D}$ is a Dynkin system, containing a $\cap$-stable generator, which implies by the Theorem of Dynkin that $\mathcal{R} = \mathcal{D}$.

(iii) implies now that all measurable, simple and non negative functions f are in F^+, and since every non negative measurable function f is the limit of an increasing sequence of measurable, simple and non negative functions, we deduce from (iv) the claim.
(b) clear

□

Often it might not possible be to verify (ii), in particular if being in F depends on the values certain integrals (like in Tonelli's and Fubini's Theorem). Thus, it might involve the difference of two sets which both might have measure ∞. In the σ-finite case we can circumvent that problem.

Theorem 4. *(Monotone Class Theorem for L_0^+ and L_1, σ-finite version)*
Let $(Z, \mathcal{R}, \rho)$ be a σ-finite measure space and let $\mathcal{E}$ be a $\cap$-stable generator of $\mathcal{E}$ containing a sequence (E_n) with $E_n \nearrow X$, and $\rho(E_n) < \infty$, for $n \in \mathbb{N}$.

Let $F^+ \subset L_0^+$ have the following properties.

i) $\{\chi_E : E \in \mathcal{E}\} \subset F^+$,
ii) *If $A, B \in \mathcal{R}$, $A \subset B$, with $B \subset E_n$ for some $n \in \mathbb{N}$ (thus $\mu(B) < \infty$) and χ_A, $\chi_B \in F^+$, then $\chi_B - \chi_A \in F^+$,*
iii) *If $f, g \in F^+$ and $a, b \ge 0$ then $af + bg \in F^+$,*
iv) *if $(f_n) \subset F$ and $f_n \nearrow f$, then $f \in F^+$.*

Then $F^+ = L_0^+$.

Proof. We first fix $n \in \mathbb{N}$ and consider

$$\mathcal{D}_n = \{A \in \mathcal{R} : A \subset E_n \text{ and } \chi_A \in F^+\}$$

Note that $\mathcal{E}_n = \{E_n \cap A : A \in \mathcal{E}\}$ is a subset of $\mathcal{E}$ (since $\mathcal{E}$ is $\cap$-stable) and generates $\mathcal{R}_n = \{E_n \cap A : A \in \mathcal{R}_n\}$ which is a σ algebra on E_n.

Following the same arguments as in the proof of Theorem 3 it follows that $\mathcal{D}_n$ is a Dynkin system, and thus, by Dynkin's Theorem, $\mathcal{D}_n = \mathcal{R}_n$. Since for any $A \in \mathcal{R}$, it follows that $A \cap E_n \nearrow A$, it follows from (iv) that $\chi_A \in F$ for all $A \in \mathcal{R}$.

Then we can again follow the same arguments as before: (iii) implies that all measurable, simple and non negative functions are in F^+. Since every f is the limit of a sequence of measurable, simple and non negative functions, we deduce the claim.

□

Using Theorem 4 we can prove Theorem 2

Proof of Theorem 2. We first make the following easy observations:
Let f, g, g_n, $n \in \mathbb{N}$, be functions on $X \times Y$, $ab \in \mathbb{R}$, $x \in X$ and $y \in Y$, then $(af + bg)_x = af_x + bg_x$, $(af + bg)_y = af_y + bg_y$, and if g is the point wise limit of the g_n's then g_x and g_y are the point wise limits of $(g_n)_x$ and $(g_n)_y$, respectively.

(a) We let F to be all the functions $f \in L_0^+$, with the following properties:

(1) f_x is $\mathcal{N}$ measurable, for all $x \in X$,

(1') f_y is $\mathcal{M}$ measurable, for all $y \in Y$,

(2) $\phi_f : X \to \mathbb{R} \cup \{\infty\}, \quad x \mapsto \int f_x(y) d\nu$ is $\mathcal{M}$-mble function

(2') $\psi_f : Y \to \mathbb{R} \cup \{\infty\}, \quad y \mapsto \int f_y(x) d\mu$ is $\mathcal{N}$-mble function

(3) $$\int f(x, y) d(\mu \otimes \nu) = \int \phi_f(x) d\mu = \int \psi_f(y) d\nu$$

We let $\mathcal{E} = \{A \times B : A \in \mathcal{M}, B \in \mathcal{N}\}$ which is a $\cap$-stable generating system, and since μ and ν are σ-finite, $\mathcal{E}$, for $n \in \mathbb{N}$ let $E_n = E_n^X \times E_n^Y$, where $E_n^X \nearrow X$ and $E_n^X \in \mathcal{M}$, with $\mu((E_n^X) < \infty$, and $E_n^Y \nearrow Y$ and $E_n^Y \in \mathcal{N}$, with $\mu(E_n^Y) < \infty$ (thus $E_n \nearrow X \times Y$ and $(\mu \otimes \nu)(E_n) < \infty$, $n \in \mathbb{N}$). Also for

$f = \chi_{A\times B}$, with $A \in \mathcal{M}$ and $B \in \mathcal{N}$, $x \in X$ and $y \in Y$

$$f_x = \chi_A(x)\chi_B = \begin{cases} \chi_B & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

$$f_y = \chi_B(y)\chi_A = \begin{cases} \chi_A & \text{if } y \in B, \\ 0 & \text{if } y \notin B. \end{cases}$$

More over

$$\phi_f = \nu(B)\chi_A \text{ and } \psi_f = \mu(A)\chi_B.$$

Thus, condition (i) of Theorem 4 follows.

We note that all conditions are closed under taking finite positive linear combinations. It is also closed under taking limits of monotone increasing sequences (such a limit satisfies limit (1) and (1') by the remarks at the beginning of the proof and (2), (2') and (3) by the Monotone Convergence Theorem). We therefore conclude that (iii) and (iv) of Theorem 4 are satisfied.

Finally, in order to verify (ii), assume that $C \subset D \subset E_n$, $\chi_C, \chi_D \in F$. Put $f = \chi_C$ and $g = \chi_D$. This implies that $0 \le f_x \le g_x \le \big(\chi_{E_n}\big)_x = \chi_{E_n^Y}$, for all $x \in X$, and thus

$$\int f_x d\nu \le \int g_x d\nu \le \nu(E_n^Y) < \infty,$$

and similarly

$$\int f_y d\nu \le \mu(E_n^X) < \infty, \text{ for all } y \in Y.$$

We deduce that $(\chi_{D\setminus C})_x = g_x - f_x \in L_1(\nu)$, for all $x \in X$, and thus

$$\phi_{g-f}(x) = \int g_x d\nu - \int f_x d\nu = \phi_g(x) - \phi_f(x),$$

is well defined for all $x \in X$, and a $\mathcal{M}$-mble and non negative function. Similarly we show that

$$\psi_{g-f}(y) = \int g_y d\mu - \int f_y d\mu = \phi_g(y) - \phi_f(y),$$

is well defined for all $y \in Y$, and a $\mathcal{N}$-mble and non negative function. Moreover

$$\begin{aligned} \mu\otimes\nu(D\setminus C) &= \mu\otimes\nu(D) - \mu\otimes\nu(C) \\ &= \int \phi_g d\mu - \int \phi_f d\mu = \int \phi_{g-f} d\mu \\ &= \int \psi_g d\nu - \int \psi_f d\nu = \int \psi_{g-f} d\nu. \end{aligned}$$

This finishes the verification of (ii) in Theorem 4 and thus the proof of part (a).

(b) Let $f \in L_1(\mu\otimes\nu)$, and thus, f^+, f^- satisfy the conclusion of part (a). Thus

$$\int \phi_{f^+}\mu = \int \Big[\int (f^+)_x(y)d\nu\Big]d\mu = \int f(x,y)d(\mu\otimes\nu) < \infty.$$

It therefore follows that

$$A^+ = \{x \in X : \phi_{f^+} = \infty\}$$

is a measurable μ null set. Similarly

$$A^- = \{x \in X : \phi_{f^-} = \infty\}$$

is a measurable μ-null set. Thus we define

$$\phi_f = \begin{cases} \phi_{f^+} - \phi_{f^-} = \int (f^+)_x(y) - f_x^-(y)d\nu & \text{if } x \notin A^+ \cup A^-, \\ 0 & \text{if } x \in A^+ \cup A^-. \end{cases}$$

Similarly, we can put

$$B^+ = \{y \in Y : \psi_{f^+} = \infty\} \text{ and } B^- = \{y \in Y : \psi_{f^+} = \infty\}$$

and define

$$\psi_f = \begin{cases} \psi_{f^+} - \psi_{f^-} = \int \Big[\int (f^+)_y(x) - f_y^-(x)d\nu\Big] & \text{if } y \notin B^+ \cup B^-, \\ 0 & \text{if } y \in B^+ \cup B^-. \end{cases}$$

We deduce from part (a) that ϕ_f is $\mathcal{M}$-mble , and ψ_f is $\mathcal{N}$-mble, and that

$$\int_f (x,y)d(\mu\otimes\nu) = \int \phi_f d\mu = \int \psi_f d\nu.$$

$\square$