1 (b) / page 78

define

$f(x) = e^x - \cos x - 1$

$f(0) = 1 - 1 - 1 = -1$ $\quad f\left(\frac{3\pi}{2}\right) = e^{\pi/2} - (-1) - 1 = e^{\pi/2}$

Since $f(x)$ is continuous, by IVT there is an $x \in [0, \pi/2]$ so that $f(x) = 0$

1 (c) / page 78

define $f(x) = 2^x - 2 + x$

$f(0) = -1 \quad f(1) = 1$

Since $f(x)$ is continuous, by IVT there is an $x \in [-1, 1]$ so that $f(x) = 0$

2 (b) let $x \in [0, 1]$

$\lim_{z \to x} 1 - \cancel{z} = \lim_{z \to x} 1 - \lim_{z \to x} \cancel{z} = 1 - x$

$\lim_{z \to x} 1 + z = \lim_{z \to x} 1 + \lim z = 1 + x$

for $x \in [0, 1]$ $\quad 1 + x \neq 0$

thus $\lim_{z \to x} \frac{1 - z}{1 + z} = \frac{1 - x}{1 + x}$

2 (c) We only check $x=0$
($x \neq 0$ straight forward)

$f(0) = 0$

for $0 < x \leq 1$

$$-\sqrt{x} \leq f(x) = \sqrt{x} \sin \frac{1}{x} \leq \sqrt{x}$$

$$\lim_{x \to 0} \pm\sqrt{x} = 0$$

Thus by Squeeze Theorem

$$\lim_{x \to 0} f(x) = 0 = f(0)$$

2 (e)

$$\lim_{z \to x} x^2 + x - 6 = \left(\lim_{z \to x} \cancel{z}\right)^2 + \lim_{z \to x} z - 6$$

$$= x^2 + x - 6 \neq 0$$

if $x \in [0,1]$

$$\lim_{z \to x} \sin(e^{z}) = \sin \lim_{z \to \cancel{x}} e^{\cancel{z}}$$

$\uparrow$ $\sin(\cdot)$ is continuous

$$= \sin\left(e^{\lim_{z \to x} z}\right) = \sin e^{x}$$

$\uparrow$ $e^{(\cdot)}$ is continuous

$$\Rightarrow \lim_{z \to \cancel{x}} \frac{\sin(e^{z})}{z^2 + \cancel{z} - 6} = \frac{\sin(e^{x})}{x^2 + x - 6}$$

(3) By extrem value theorem
there is an x_1
so that $f(x_1) = \sup_{x \in [a,b]} f(x)$

and an x_2
so that $f(x_2) = \inf_{x \in [a,b]} f(x)$

Thus $f(x_2) \le f(x) \le f(x_1)$ for all $x \in [a,b]$

Thus $|f(x)| \le \max(|f(x_2)|, |f(x_1)|)$
for all $x \in [a,b]$

Thus $\sup_{x \in [a,b]} |f(x)| < \infty$

(6) (a) continuous for $x \neq 0$: straight forward

dis continuous at 0
Need to show
$\exists_{\varepsilon > 0} \forall_{\delta > 0} \exists_{x \in [-\delta, \delta]} \quad |f(x) - f(0)| \geq \varepsilon$

Choose $\varepsilon = 1$
Let $\delta > 0$ choose $x = \frac{1}{2\pi n}$ with $2\pi n > \frac{1}{\delta}$
Thus $x \in [-\delta, \delta]$
and $f(x) = \cos 2n\pi = 1$
$|f(x) - f(0)| = |f(x)| = 1$

(b) Continuity at $x \neq 0$ follows from product rule

Continuity at $x = 0$:

For $0 \neq x$.

$$-|g(x)| \leq f(x)\cdot g(x) \leq |g(x)| \quad [\text{since } |f(x)| \leq 1]$$

$$\underset{-c\sqrt{x}}{\rotatebox{90}{$\leq$}} \qquad\qquad \underset{c\sqrt{x}}{\rotatebox{90}{$\geq$}}$$

thus by Squeeze Theorem

$$\lim_{x\to 0} f(x)\cdot g(x) = 0 = f(0)\,g(0)$$

7 (a) "$\Rightarrow$" Assume $g(x)$ continuous at a then

$$\lim_{x\to a} f(x)+g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$= f(a) + g(a)$$

thus $g(x)+f(x)$ continuous at a

"$\Leftarrow$" Assume $g(x)+f(x)$ continuous at a

$$\lim_{x\to a} g(x) = \lim_{x\to a} g(x) + f(x) - f(x)$$

$$\left\{\begin{array}{l} = \lim_{x\to a} g(x)+f(x) - \lim_{x\to a} f(x) \\ = g(a)+f(a) - f(a) = g(a) \end{array}\right.$$

Thus $g(x)$ cont. at a

(b) Statement should be
Assume $f(x)$ continuous at a
$\boxed{\text{and } f(a) \neq 0}$ then

$f(x)\,g(x)$ cont. at a
$\iff$ $g(x)$ cont at a

Proof similar to part (a)

Why is $\boxed{\ldots}$ necessary:
take
$$g(x) = \begin{cases} \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f(x) = x$$

then $f(x) \cdot g(x)$ continuous at 0

but $g(x)$ <u>not</u> cont. at 0
{see previous problem}

10) Let $A = f(0)$
By assumption there is
a $C > 0$ so that
$f(x) \geq |A|$ for all $x > C$
$f(x) \geq |A|$ for all $x < C$

By Extreme Value Theorem

f has Minimum x_m on $[-C, C]$

Note that $f(x_m) \leq f(0) \leq A \leq |A|$

and that $f(x) \geq |A|$ for all $x \notin [-C, C]$

thus x_m is Minimum of $f(x)$
on all of $(-\infty, \infty)$