Solution, Homework 8

page 83 / 1(c) $\quad f(x) = x \sin 2x \quad$ is uniformly continuous

For $x, y \in \mathbb{R}$ note

$$|f(x) - f(y)| = |x \cdot \sin 2x - y \sin 2y|$$

$$\le |x \cdot \sin 2x - y \sin 2x| + |y \sin 2x - y \sin 2y|$$

$$\le |x - y| + |y|\,|\sin 2x - \sin 2y|$$

If $0 \le x, y \le 1$

$$|f(x) - f(y)| \le |x - y| + |\sin 2x - \sin 2y|$$

$$\le |x - y| + |2x - 2y|$$

$$= 3|x - y|$$

So if $\varepsilon > 0$ let $\delta = \frac{\varepsilon}{3}$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

2(d) define

$$g(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

then $g(x)$ is continuous function on $[0,1]$ [we proved $\lim_{x \to 0} \frac{\sin x}{x} = 1$]

Thus, by Theorem $g(x)$ is uniform continuous on $[0,1]$.

Thus $g(x)$ is uniform continuous on $(0,1)$

4 (a) Let $\varepsilon > 0$ we need to find $\delta > 0$ so that:

if $|x-y| < \delta$ then $|f(x)-f(y)| < \varepsilon$

Choose $C > 0$ so that

(*) $\forall x \geq C \quad |f(x) - L| < \varepsilon/2$

Choose $\delta > 0$ so that [use Theorem of class]

(**) $|x-y| < \delta$, $x, y \in [0, C] \Rightarrow |f(x) - f(y)| < \varepsilon/2$

Now it follows for any $x, y \in \mathbb{R}$ with $|x-y| < \delta$

Case 1 $\quad x, y \in [0, C] \quad |f(x) - f(y)| < \varepsilon$ by (**)

Case 2 $\quad x, y \in [C, \infty) \quad |f(x) - f(y)| \leq |f(x) - L| + |L - f(y)| < \varepsilon$

Case $\quad x \leq C \leq y \quad |f(x) - f(y)| \leq |f(x) - f(C)| -$

Case 3 $\quad x \le c \le y$

$$|f(x) - f(y)| \le |f(x) - f(c)| + |f(c) - f(y)|$$
$$\le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

case 4 $\quad y \le c \le x$ as case 3

4(b) since $\lim\limits_{x \to \infty} \frac{1}{1+x^2} = 1$

4(a) can be applied

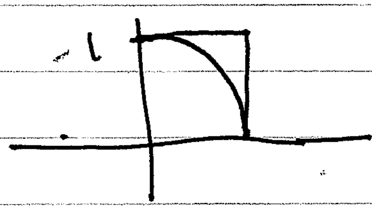
114 / 1(a) $\quad f = 1 - x^2 \quad P = \{0, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, 1\}$

Since f is decreasing on $[0,1]$

$$U(f, P) = f(0)\left(\tfrac{2}{5} - 0\right) + f\left(\tfrac{2}{5}\right)\left(\tfrac{1}{2} - \tfrac{1}{5}\right) + f\left(\tfrac{1}{2}\right)\left[\tfrac{3}{5} - \tfrac{1}{2}\right] + f\left(\tfrac{3}{5}\right)\left[1 - \tfrac{3}{5}\right]$$

$$L(f, P) = f\left(\tfrac{2}{5}\right)\left(\tfrac{2}{5} - 0\right) - f\left(\tfrac{1}{2}\right)\left[\tfrac{1}{2} - \tfrac{1}{5}\right] + f\left(\tfrac{3}{5}\right)\left[\tfrac{3}{5} - \tfrac{1}{2}\right] + f(1)\left[1 - \tfrac{3}{5}\right]$$

Picture



$U(f,P)$ is better approximation because f is concave down

Then choose n_0 so that

$$\left| \frac{M}{n_0} \cdot \ell \right| \leq \varepsilon/2 \quad \text{where } M = \sup_{x \in [0,1]} |f(x)|$$

Then for $n \geq n_0$

$$U(f, P_n) - I_0$$

$$\leq U(f, P_n') - I_0 \quad [P_n' = P \cup P_n]$$
$$+ \ell \cdot M \, \|P_n\|$$

$$\leq U(f, P) - I_0 + \varepsilon/2 \leq \varepsilon$$

Similar with $L(f, P_n)$