Practice Test for Final (Chapters 1,2, 3, 4(1), and 5), Math220

General Comment: The final will be cumulative. One third of the problems, more or less, will come from Chapters 1-3, while two thirds from Chapters 4(only section 4.1) and 5.

(1) Write the following statements using (only) symbols.
   (a) Every real number has a fifth root.
   (b) For any two integers a and b, where a is not 0, there are unique integers q and r so that r is between 0 and $|a|-1$ and $b = aq + r$.

(2) Negate and write only in symbols not using $\neg$: Every real number has a square root.

(3) Let A and B be sets and let $(A_i : i \in I)$ be a family of sets. Define the following:
   (a) The intersection of A and B,
   (b) the union of A and B,
   (c) the intersection of $(A_i : i \in I)$,
   (d) the union of $(A_i : i \in I)$,
   (e) the Cartesian product of A and B,
   (f) the difference of A and B.

(4) Assume that $(A_i : i \in I)$ is a family of subsets of a set X (complements are taken in X). Prove

$$\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A_i} \text{ and } \overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$$

(5) For a set A define the power set $P(A)$.

(6) Let $f : A \to B$ be function from A to B and let $X \subseteq A$, and $Y \subseteq B$, define
   (a) $f^{-1}(Y)$,
   (b) $f(X)$.

(7) Let $f : A \to B$ be function from A to B, define what it means, that
   (a) f is injective,
   (b) f is surjective,
   (c) f is bijective,
   (d) f is invertible.

(8) Let $f : A \to B$, and X, Y be subsets of A. Prove:
   (a) $f(X \cap Y) \subseteq f(X) \cap f(Y)$. Find an example for which we have $f(X \cap Y) \subsetneq f(X) \cap f(Y)$
   (b) $f(X \cup Y) = f(X) \cup f(Y)$,

(c) If $X \subseteq Y$ then $f(X) \subseteq f(Y)$,
(d) $X \subset f^{-1}(f(X))$ and find an example for which we have $f(f^{-1}(X)) \subsetneq X$.

(9) What do we mean by a *binary operation on a set* A?

(10) For a binary operation $*$ on a set A define what it means that:
(a) $*$ is *associative*,
(b) $*$ is*commutative*,
(c) $*$ *has a neutral element*,
(d) an $a \in A$ has *an inverse*.

(11) Assume that $*$ is a binary operation on A with a neutral element. Then this neutral element must be unique.

(12) Assume that $*$ is an associative binary operation on A with a neutral element 0. Show that if $a \in A$ has an inverse, this inverse must be unique.

(13) Let $M_2(\mathbb{R})$ be the 2 by 2 matrices with the operation $\circ$ defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

is associative, not commutative, and has a neutral element.

Show that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has an inverse if and only if $ad - bc \neq 0$.

(14) Let A be a set of at least 2 elements and $F(A)$ all functions from A to A with the composition as binary operation. Decide if $F(A)$ is associative, commutative, has a neutral element, and has for all elements an inverse.

(15) Let $*$ be a binary operation on the set A let $B \subset A$. Define what it means that B *is closed under* $*$.
Decide, with proof, whether the sets

$$B = \{f : \mathbb{R} \to \mathbb{R} : f(3) = 3\}, \quad C = \{f : \mathbb{R} \to \mathbb{R} : f(3) = 2\}, \text{ and}$$
$$L = \{f : \mathbb{R} \to \mathbb{R} : f(x) = ax + b, \text{ for some } a, b \in \mathbb{R}\}$$

are closed under $\circ$ in $F(\mathbb{R})$.

(16) Define on a set with two elements a binary operation which is not associative.

(17) Problem 2 on page 134.

(18) State Axioms A1-A8 for the integers $\mathbb{Z}$.

(19) Define on $\{0, 1\}$ and on $\{0, 1, 2\}$ addition and multiplication so that A1-A8 are satisfied.

(20) Only using A1-A8, and results you deduced from A1-A8, prove the following $a, b, c \in \mathbb{Z}$ (first state what these equalities mean)
  (a) (Cancellation law) $a + c = b + c \Rightarrow a = b$,
  (b) $-(-a) = a$,
  (c) $(-a)b = a(-b) = -(ab)$,
  (d) $(-1)(-1) = 1$,
  (e) $(-1)a = -a$,
  (f) $a(b - c) = ab - ac$ (what is "$a - b$"?)
  (g) $0a = a0 = 0$.

(21) For $n \in \mathbb{Z}^+$ define on $\{0, 1, 2, \dots n - 1\}$ addition and multiplication so that A1-A8 are satisfied.

(22) State Axioms A9-A10 for the integers $\mathbb{Z}$.

(23) Consider the addition and multiplication on $\{0, 1, 2\}$ you defined in (19), and prove that you cannot find a subset A^+ of $A = \{0, 1, 2\}$, so that A9 and A10 are satisfied.

(24) We define for $a, b \in \mathbb{Z}$

$$a < b :\iff b - a \in \mathbb{Z}^+.$$

Using this definition and A1-A10, show for $a, b, c \in \mathbb{Z}$:
  (a) $a < b \vee b < a \vee a = b$, and only one holds,
  (b) $a > 0 \Rightarrow -a < 0$,
  (c) $-1 < 0 < 1$,
  (d) $a < b$, $c > 0 \Rightarrow ac < bc$,
  (e) $a < 0, b < 0 \Rightarrow ab > 0$,
  (f) $ab = 0 \Rightarrow a = 0 \vee b = 0$,
  (g) If $1 < a, b$ then $ab > \max(a, b)$.

(25) State the Well-Ordering principle A11.

(26) Using the Well-Ordering principle show that there is no $a \in \mathbb{Z}$ so that $0 < a < 1$ (See Proposition 5.1.6).

(27) Give examples of number systems satisfying A1-A10, which do **not** satisfy A11.

(28) State the *First Principle of Mathematical Induction* and prove using A11.

(29) State the modified form of the first principle of Mathematical Induction, and prove it.

(30) State the *Second Principle of Mathematical Induction* and prove it using A11.

(31) Prove the following for all $n \in \mathbb{Z}^+$:
  (a) $2^n > n$,
  (b) $2^n < n!$, if $n \geq 4$,

(c) $\sum_{j=1}^{n} j = \frac{1}{2}n(n+1)$,

(d) The cardinality of the power set of a set with n elements has 2^n elements.

(e) $\sum_{j=1}^{n} a^n = \frac{a^{n+1} - a}{a - 1}$.

(f) $\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$.

(g) If f, g are two n-times differentiable functions on $\mathbb{R}$ then

$$\frac{d^n fg}{dx^n}(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{d^j f}{dx^j}(x) \frac{d^{n-j} g}{dx^j}(x).$$

(32) Let $a, b \in \mathbb{Z}$. What do we mean by $a|b$?

(33) Define what it means that $a, b \in \mathbb{Z}$ *are relatively prime*

(34) State the Theorem on the Division Algorithm for Integers (Theorem 5.3.1). Go through the proof.

(35) Let $a, b \in Z$. What do we mean by *the greatest common divisor of a and b*? We will denote the common greatest divisor of a and b by $\gcd(a, b)$ (note: book uses (a, b) instead of $\gcd(a, b)$).

(36) Show that for two numbers $a, b \in \mathbb{Z}$, which are not both 0 there is only one greatest common divisor and:

$$\gcd(a, b) = \min\{d \in \mathbb{Z}^+ : \exists x, y \in \mathbb{Z} \quad d = xa + yb\}$$

(see proof of Theorem 5.3.5) .

(37) Show for $a, b \in Z$:

$$a, b \text{ relative prime} iff \exists x, y \in \mathbb{Z} \quad xa + yb = 1.$$

(38) State *the Euclidean Algorithm* (Lemma 5.3.6)

(39) Find $\gcd(9180, 1122)$.

(40) Prove that $4|5^n - 1$ for $n \in \mathbb{Z}^+$,
Prove that $7|2^3 n - 1$, for $n \in \mathbb{Z}^+$

(41) Write $n \in \mathbb{Z}^+$ in its decimal form:

$$n = \sum_{j=0}^{k} r_j 10^j,$$

and show that

$$3|n \iff 3|\sum_{j=0}^{k} r_j.$$

(42) Prove for $n \in \mathbb{Z}$: $\gcd(n, n+1) = 1$.

(43) Define what is a *prime number*. Define what is a composite numbers.

(44) Prove the following steps of the Unique Factorization Theorem:
   (a) If n is a positive composite number then there are numbers $a, b \in \mathbb{Z}$ with $1 < a, b < n$ and $ab = n$.
   (b) Show that in (a) a can be taken so that $a < \sqrt{n}$.
   (c) Show that any $a \in \mathbb{Z}$, with $a > 1$ is divisible by a prime number.
   (d) Show that if a prime number p divides ab, for some $a, b \in \mathbb{Z}$ it divides a or b.
   (e) Show that if a prime number p divides $a_1, a_2, \dots a_\ell$, where $a_1, a_2, \dots, a_\ell \in \mathbb{Z}$, then it divides$a_i$ for some $i \leq \ell$.

(45) State and prove the *Unique Prime Factorization Theorem*

(46) Prove that there are infinitely many prime numbers.