A PROBLEM ON SPREADING MODELS

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Abstract. It is proved that if a Banach space X has a basis (e_n) satisfying every spreading model of a normalized block basis of (e_n) is 1-equivalent to the unit vector basis of ℓ_1 (respectively, c_0) then X contains ℓ_1 (respectively, c_0). Furthermore Tsirelson's space T is shown to have the property that every infinite dimensional subspace contains a sequence having spreading model 1-equivalent to the unit vector basis of ℓ_1. An equivalent norm is constructed on T so that $\|s_1+s_2\|<2$ whenever (s_n) is a spreading model of a normalized basic sequence in T.

§0. Introduction.

From the fact that ℓ_1 (and c_0) are not distortable [J] it follows that if a Banach space X contains ℓ_1 (or c_0) then some basic sequence (e_i) in X has the property that every spreading model of a normalized block basis is 1-equivalent to the unit vector basis of ℓ_1 (or c_0). In this paper we prove the converse statements. More generally we show

Theorem A. *Let (e_i) be a basis for X*

a) *If $\|s_1+s_2\|=2$ whenever (s_n) is any spreading model of a normalized block basis of (e_i), then X contains a subspace isomorphic to ℓ_1.*

b) *If $\|s_1+s_2\|=1$ whenever (s_n) is any spreading model of a normalized block basis of (e_i), then X contains a subspace isomorphic to c_0.*

The proof of a) will be achieved by showing that such an (e_i) cannot be weakly null, if normalized. The proof will depend heavily on the theory of the generalized Schreier classes of subsets of $\mathbb{N}$ as introduced in [AA]. We make strong use of recent results in [AMT] as well as a result from [AO].

From Theorem A we obtain the

Research supported by NSF and TARP.

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Corollary. *Let* (e_i) *be a basis for* X*. If* X *contains no subspace isomorphic to* ℓ_1 *or* c_0 *then there exists a normalized block basis of* (e_i) *having spreading model* (s_i) *satisfying*

$$1 < \|s_1 + s_2\| < 2 \ .$$

Theorem A a) is proved in §2 while part b) and the (easy) Corollary are proved in §3.

In conjunction with Theorem A it is worth considering Tsirelson's space T. T is reflexive with an unconditional basis (t_i) and yet all spreading models of normalized block bases of (t_i) are 2-equivalent to the unit vector basis of ℓ_1. Furthermore we have

Theorem B. *Let* X *be an infinite dimensional subspace of* T*. Then there exists* $(x_i) \subseteq X$ *with spreading model 1-equivalent to the unit vector basis of* ℓ_1*.*

We prove this theorem in §4. Furthermore we show that T can be renormed to fail the conclusion of Theorem B.

We do not know if Theorem A can be extended to ℓ_p $(1 < p < \infty)$.

Problem. Let (e_i) be a basis for X and $1 < p < \infty$. Suppose that every spreading model of any normalized block basis of (e_i) is 1-equivalent to the unit vector basis of ℓ_p. Does X contain ℓ_p, either isomorphically or almost isometrically?

§1 Preliminaries.

Definition 1.1. Let (e_i) be a normalized basic sequence. A basic sequence (s_i) is a *spreading model* of (e_i) if for some sequence $\varepsilon_n \downarrow 0$ and all $(a_i)_1^n \subseteq [-1,1]^n$ we have

$$\left| \left\| \sum_1^n a_i e_{k_i} \right\| - \left\| \sum_1^n a_i s_i \right\| \right| < \varepsilon_n \text{ whenever } n \le k_1 < \cdots < k_n \ .$$

It is well known that every normalized basic sequence has a subsequence with a spreading model. Also (s_i) is necessarily *spreading* ($\|\sum_1^n a_i s_i\| = \|\sum_1^n a_i s_{k_i}\|$ if $k_1 < k_2 < \cdots < k_n$). If (e_i) is weakly null then (s_i) is suppression-1-unconditional

($\|\sum_F a_i s_i\| \le \|\sum_1^n a_i s_i\|$ if $F \subseteq \{1,\dots,n\}$). These and other results on spreading models can be found in [BL].

$[\mathbb{N}]$ denotes the set of all subsequences of $\mathbb{N}$. If $M \in [\mathbb{N}]$, $[M]$ is the set of all subsequences of M. $[M]^{<\omega}$ is the class of all finite subsets of M. If $E, F \in [\mathbb{N}]^{<\omega}$, "$E < F$" means that $\max E < \min F$, "$k < E$" means $\{k\} < E$.

Definition 1.2. [AA] **Generalized Schreier classes**. The classes $(S_\alpha)_{\alpha<\omega_1}$ of collections of finite subsets of $\mathbb{N}$ are inductively defined as follows

$$S_0 = \big\{\{n\} : n \in \mathbb{N}\big\}$$

$$S_{\alpha+1} = \Big\{E : E = \bigcup_1^k E_i \text{ for some } k \in \mathbb{N} \text{ and}$$

$$k \le E_1 < \cdots < E_k \text{ where } E_i \in S_\alpha \text{ for } i \le k\Big\}$$

If α is a limit ordinal, choose $\alpha_n \uparrow \alpha$ and set $S_\alpha = \{E : k \le E \in S_{\alpha_k}$ for some $k \in \mathbb{N}\}$. We also consider the empty set $\emptyset \in S_\alpha$ for all α. The definition of S_α depends upon this particular choice of (α_n) but the results we use concerning the S_α's are independent of that choice.

The Schreier classes have played a prominent role in a number of recent papers (e.g., [AA], [AD], [AMT], [OTW], [AO]).

If $M = (m_i) \in [\mathbb{N}]$ and $\alpha < \omega_1$, $S_\alpha(M) = \{(m_i)_{i\in F} : F \in S_\alpha\}$. It is easy to see that $S_\alpha(M) \subseteq S_\alpha$. We also recall that the classes S_α (or $S_\alpha(M)$) are all *regular*. By this we mean they are pointwise closed, *hereditary* ($E \subseteq F \in S_\alpha \Rightarrow E \in S_\alpha$) and *spreading* ($(n_i)_1^k \in S_\alpha$ and $m_1 < \cdots < m_k$ with $m_i \ge n_i$ for $i \le k$ implies that $(m_i)_1^k \in S_\alpha$).

Proposition 1.3 [AO]**.** *Let $N \in [\mathbb{N}]$. There exists $M \in [N]$ such that for all $\alpha < \omega_1$ if $F \in S_\alpha$ and $F \subseteq M$ then $F \setminus \min(F) \in S_\alpha(N)$.*

We also need some definitions and a result from [AMT]. Let

Definition 1.4 [AMT]**.** If $M = (m_i) \in [\mathbb{N}]$ and (e_i) is a normalized basic sequence we inductively define $\alpha_n^M = \alpha_n^M(e_i)$ for any ordinal $\alpha < \omega_1$ and $n \in \mathbb{N}$ as follows

$$0_n^M = e_{m_n}.$$

Assuming that $(\alpha)_n^M$ has been chosen we let

$$(\alpha+1)_1^M = \frac{1}{m_1}\sum_{i=1}^{m_1}\alpha_i^M$$

and assuming that

$$(\alpha+1)_i^M$$

has been chosen for $i=1,\dots,n$ we let $k_n = \min\{k\in\mathbb{N} : m_k > \operatorname{supp}(\alpha)_n^M\}$ and put

$$(\alpha+1)_{n+1}^M = \frac{1}{m_{k_n}}\sum_{i=k_n}^{k_n+m_{k_n}-1}\alpha_i^M .$$

If $\alpha = \lim \alpha_n$ is a limit ordinal we set

$$\alpha_1^M = (\alpha_{m_1})_1^M \quad \text{and for} \quad k>1\ , \quad \alpha_k^M = (\alpha_{n_k})_1^{M_k}$$
$$\text{where} \quad M_k = \{m\in M : m > \operatorname{supp}\alpha_{k-1}^M\} \text{ and } n_k = \min M_k\ .$$

Proposition 1.5 [AMT].

1) *For $\alpha<\omega_1$, $(\alpha_n^M) = (\alpha_n^M(e_i))_{n=1}^\infty$ is a convex block basis of (e_i) and $\bigcup_n \operatorname{supp}(\alpha_n^M) = M$. Moreover $\operatorname{supp}\alpha_n^M \in S_\alpha$ for all n.*
2) *If $M\in[\mathbb{N}]$, $\alpha<\omega_1$ and $(n_k)\in[\mathbb{N}]$ then $\alpha_{n_k}^M = \alpha_k^{M'}$ where $M' = \bigcup_k \operatorname{supp}(\alpha_{n_k}^M)$.*

If $x=\sum a_i e_i \in \langle e_i\rangle$ and $F\subseteq\mathbb{N}$ we define $\langle x, F\rangle = \sum_{i\in F} a_i$.

Definition 1.6 [AMT]. Let $\mathcal{F}$ be an hereditary collection of subsets of $\mathbb{N}$. Let $M\in[\mathbb{N}]$, $\varepsilon>0$, $\alpha<\omega_1$ and let (e_i) be a normalized basic sequence. $\mathcal{F}$ is (M,α,ε) *large* if for all $N\in[M]$ and $n\in\mathbb{N}$ for $\alpha_n^N = \alpha_n^N(e_i)$,

$$\sup_{F\in\mathcal{F}}\langle \alpha_n^N, F\rangle \ge \varepsilon\ .$$

Theorem 1.7 [AMT, Proposition 2.3.2 and Theorem 2.2.6]. *If $\mathcal{F}$ is (M,α,ε) large then there exists $N\in[M]$ with*

$$\mathcal{F}\supseteq S_\alpha(N)\ .$$

§2. ℓ_1 spreading models.

In this section we prove

Theorem 2.1. *Let* (e_i) *be a normalized basis for* X *having the property that* $\|s_1+s_2\|=2$ *whenever* (s_n) *is a spreading model of* (x_i) *where* $x_i=y_i/\|y_i\|$ *and* (y_i) *is any convex block subsequence of* (e_i) *satisfying* $\lim_i\|y_i\|>0$. *Then* (e_i) *is not weakly null. Moreover for all* $\varepsilon>0$ *there exists* $M=(m_i)\in[\mathbb{N}]$ *and* $x^*\in S(X^*)$ *with* $x^*(e_{m_i})>1-\varepsilon$ *for all* i.

From Theorem 2.1 it follows that in a space X whose block bases have only ℓ_1 as spreading model no block basis is weakly null. But this implies that in such a space no block basis can be weakly Cauchy. Thus in light of Rosenthal's theorem [R], Theorem A a) is a quick consequence of Theorem 2.1.

The hypothesis yields that for all n, $\|\sum_1^{2^n}s_i\|=2^n$ from which it follows that $\|\sum_1^k s_i\|=k$ for all k. We shall use this below in the following way. Given $\varepsilon>0$ there exists a subsequence (x_{n_k}) of (x_i) so that for all k, $\frac{1}{r_k}\|x_{n_k}+x_{n_{k+1}}+\cdots+x_{n_{k+r_k-1}}\|>1-\varepsilon$ where $r_k=\min(\operatorname{supp}(x_{n_k}))$, w.r.t. (e_i).

Proof. Given $\varepsilon>0$ set

$$\mathcal{F}_\varepsilon=\{F\subseteq\mathbb{N}:\ \text{there exists } x^*\in S(X^*) \text{ with } x^*(e_i)>1-\varepsilon \text{ for } i\in F\}\ .$$

We shall prove by induction on α that (P_α) holds for all $\alpha<\omega_1$ where

$$(P_\alpha)\qquad\begin{cases}\text{For all } M\in[\mathbb{N}] \text{ and } \varepsilon>0 \text{ there exists}\\ N\in[M] \text{ with } \mathcal{F}_\varepsilon\supseteq S_\alpha(N)\ .\end{cases}$$

(P_0) is clear. If α is a limit ordinal and S_α is defined via the sequence $\alpha_n\uparrow\alpha$ we proceed as follows. Given M and $\varepsilon>0$ we can choose, by the inductive hypothesis, $M\supseteq N_1\supseteq N_2\supseteq\cdots$ so that $\mathcal{F}_\varepsilon\supseteq S_{\alpha_n}(N_n)$ for $n\in\mathbb{N}$. Let $N_n=(k_i^n)_{i=1}^\infty$ and set $N=(k_n^n)_{n=1}^\infty$. Then $\mathcal{F}_\varepsilon\supseteq S_\alpha(N)$.

Finally assume that (P_β) holds and let $M\in[\mathbb{N}]$ and $\alpha=\beta+1$. Let $\varepsilon>0$ and choose $\varepsilon'>0$ so that $\varepsilon'<\varepsilon/2$. We may assume that $\mathcal{F}_{\varepsilon'}\supseteq S_\beta(M)$.

Claim. There exists $N\in[M]$ so that $\mathcal{F}_{3\varepsilon}$ is (N,α,ε) large.

Indeed for $n \in \mathbb{N}$ define

$$\mathcal{A}_n = \left\{ L \in [M] : \sup_{F \in \mathcal{F}_{3\varepsilon}} \langle \alpha_n^L, F \rangle > \varepsilon \right\} .$$

Then $\mathcal{A}_n$ is a pointwise closed subset of $[\mathbb{N}]$ and so $\mathcal{A} = \bigcap_n \mathcal{A}_n$ is Ramsey (see [E], also [O]). Thus there exists $N \in [M]$ with either $[N] \subseteq \mathcal{A}$ or $[N] \subseteq [\mathbb{N}] \setminus \mathcal{A}$. By passing to a subsequence of N we may assume by Proposition 1.3, that for $\gamma < \omega_1$ if $F \in S_\gamma$, $F \subseteq N$ then $F \setminus \{\min(F) \in S_\gamma(M)\}$. Furthermore we may assume that for all n, $\|\beta_n^L\|_\infty < \varepsilon'$ for all $L \in [N]$. (Indeed this holds if $n_1 = \min N$ satisfies $n_1^{-1} < \varepsilon'$.) It follows that for all n, since $\operatorname{supp} \beta_n^N \setminus \min(\operatorname{supp} \beta_n^N) \in S_\beta(M) \subseteq \mathcal{F}_{\varepsilon'}$, that $\|\beta_n^N\| > 1 - 2\varepsilon'$. ¿From the hypothesis of our theorem applied to $x_n = \beta_n^N / \|\beta_n^N\|$ we obtain a subsequence $(\beta_{n_k}^N)_{k=1}^\infty$ satisfying for all k:

$$(*) \qquad \frac{1}{r_k} \|\beta_{n_k}^N + \beta_{n_{k+1}}^N + \cdots + \beta_{n_{k+r_k-1}}^N\| > 1 - 3\varepsilon'$$

where $r_k = \min(\operatorname{supp} \beta_{n_k}^N)$.

Let $L = \bigcup_k \operatorname{supp} \beta_{n_k}^N$. Then by Proposition 1.5, $\beta_k^L = \beta_{n_k}^N$. Hence from $(*)$ and the definition of α_n^L we have $\|\alpha_n^L\| > 1 - 3\varepsilon'$ for all n.

Let $n \in \mathbb{N}$ and $x^* \in S(X^*)$ with $x^*(\alpha_n^L) > 1 - 3\varepsilon'$. Write $\alpha_n^L = \sum_1^p a_i e_i$ and set $F = \{i : x^*(e_i) > 1 - 3\varepsilon\}$. Then $\sum_{i \in F} a_i > \varepsilon$ (otherwise $x^*(\alpha_n^L) \leq \sum_{i \in F} a_i + 1 - 3\varepsilon \leq 1 - 2\varepsilon < 1 - 3\varepsilon'$). Thus $L \in \mathcal{A}$ and hence $[N] \subseteq \mathcal{A}$, whence the claim follows. Thus by Theorem 1.7, (P_α) holds (we actually proved (P_α) for 3ε replacing ε).

Since (P_α) holds for all $\alpha < \omega_1$ we obtain the "moreover" statement of the theorem. Indeed this follows easily from an argument of Bourgain [B]. Let T be the tree $T = \{(n_i)_1^k : n_1 < \cdots < n_k$ and there exists $x^* \in S(X^*)$ with $x^*(e_{n_i}) > 1 - \varepsilon$ for $i \leq k\}$. T is a closed tree and thus if T were well founded (no infinite branches) then the order of T is $< \omega_1$. But since (P_α) holds, the order of $T \geq \omega^\alpha$ for all α. The latter holds since the order of $S_\alpha(N)$ is ω^α, as is well known (see e.g., [AA] or [OTW]). $\square$

§3. c_0 spreading models.

In this section we prove Theorem A b) and the corollary. Note that the hypothesis yields that $\|\sum_1^n s_i\| = 1$ for all n. Also the hypothesis is satisfied if all spreading models of normalized block bases of (e_i) are 1-equivalent to the unit vector basis of c_0 but this is a stronger condition than the hypothesis as the following example indicates.

Example 3.1. Let $\|\sum_1^n a_i e_i\| = \max_{i<j} |a_i - a_j|$. Then if X is the completion of $(\langle e_i\rangle, \|\cdot\|)$, X satisfies the hypothesis of the theorem yet (e_i), which is its own spreading model, is not 1-equivalent to the unit vector basis of c_0.

Assume that X has a basis (e_i) satisfying the hypothesis of b). We break the proof into several steps. For $a, b \in \langle e_i\rangle$ we write "$a < b$" if $\mathrm{supp}(a) < \mathrm{supp}(b)$.

Step 1. For all $\varepsilon > 0$ and $\ell \in \mathbb{N}$ there exists $m \in \mathbb{N}$ so that for all $a > e_m$ with $\|a\| = 1$ there exists $b_1 < \cdots < b_\ell \le e_m$, $\|b_i\| = 1$ for $i \le \ell$, such that for all $1 \le q \le p \le \ell$ and $\delta = 0$ or 1,

$$\left| \left\| \sum_q^p b_i + \delta a \right\| - 1 \right| < \varepsilon .$$

Proof. If not then there exists $\varepsilon > 0$ and $\ell \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $a_m > e_m$, $\|a_m\| = 1$ so that for all $b_1 < \cdots < b_\ell \le e_m$, $\|b_i\| = 1$ there exists $1 \le q_m \le p_m \le \ell$ with for some $\delta_m = 0$ or 1,

$$\left| \left\| \sum_{q_m}^{p_m} b_i + \delta_m a_m \right\| - 1 \right| > \varepsilon .$$

Choose a subsequence (a_{m_i}) of (a_m) having a spreading model (s_i). Since $\|\sum_1^{\ell+1} s_i\| = 1 = \|\sum_1^\ell s_i\|$, we may assume that $\big| \|\sum_q^p a_{m_i} + \delta a_{m_{\ell+1}}\| - 1\big| < \varepsilon$ for all $q \le p \le \ell$ and $\delta = 0$ or 1 and that $a_{m_\ell} < e_{m_{\ell+1}}$. This contradicts our choice of $a_{m_{\ell+1}}$.

Let $\varepsilon_i \downarrow 0$ with $\sum_1^\infty \varepsilon_i < 1$. Applying Step 1 to $\varepsilon = \varepsilon_1$ and $\ell = 2$ we obtain m_1 so that for all $a > e_{m_1}$, $\|a\| = 1$ there exist $x_1 < y_1 \le e_{m_1}$, $\|x_1\| = \|y_1\| = 1$ and

$$\big| \|x_1 + y_1 + a\| - 1 \big| < \varepsilon_1 \quad \text{and} \quad \big| \|y_1 + a\| - 1 \big| < \varepsilon_1 .$$

Step 2. There exist $1 = m_0 < m_1 < m_2 < \cdots$ such that for all k and $a > e_{m_k}$, $\|a\| = 1$, there exists $x_1 < y_1 \le e_{m_1} < x_2 < y_2 \le e_{m_2} < \cdots < x_k < y_k \le e_{m_k}$

satisfying $\big|\,\|x_i\|-1\big|<\varepsilon_i$, $\big|\,\|y_i\|-1\big|<\varepsilon_i$ for $i\le k$ and for all $F\subseteq\{1,2,\dots,k\}$

$$\Big\|\sum_F x_i+\sum_1^k y_i+a\Big\|\le 1+\sum_1^k\varepsilon_i\ .$$

Proof. We proceed by induction on k. The case $k=1$ was presented above. Assume $m_1<\cdots<m_{k-1}$ have been chosen. Let $\varepsilon>0$ satisfy $\frac{\varepsilon}{1-\varepsilon}+(2k-2)\varepsilon<\varepsilon_k$. We can, by a compactness argument, find ℓ so that if the induction hypothesis for $k-1$ is applied to each of ℓ different a's $>e_{m_{k-1}}$, say $(a_n)_1^\ell$, then if $(x_i^n)_1^{k-1}$, $(y_i^n)_1^{k-1}$ satisfy Step 2 for a_n, for some $n\ne m\le\ell$ we have $\|x_i^n-x_i^m\|,\|y_i^n-y_i^m\|<\varepsilon$ for $i\le k-1$. Choose m_k by Step 1 applied for this ℓ and ε to $(e_i)_{i=m_{k-1}+1}^\infty$. Let $a>e_{m_k}$ with $\|a\|=1$. Choose $e_{m_{k-1}}<b_1<\cdots<b_\ell\le e_{m_k}\le a$ to satisfy

$$\bigg|\Big\|\sum_q^p b_i+\delta a\Big\|-1\bigg|<\varepsilon \text{ for } 1\le q\le p\le\ell\ ,\qquad \delta=0,1\ .$$

By Step 2 (for $k-1$) there exists for $1\le q\le\ell$, $x_1^q<y_1^q\le e_{m_1}<\cdots<x_{k-1}^q<y_{k-1}^q\le e_{m_{k-1}}$ so that $\big|\,\|x_i^q\|-1\big|,\big|\,\|y_i^q\|-1\big|<\varepsilon_i$ for $i\le k-1$ and so that for $F\subseteq\{1,\dots,k-1\}$,

$$\Big\|\sum_F x_i^q+\sum_1^{k-1}y_i^q+\Big\|\sum_q^\ell b_i+a\Big\|^{-1}\Big(\sum_q^\ell b_i+a\Big)\Big\|\le 1+\sum_1^{k-1}\varepsilon_i\ .$$

Thus

$$\Big\|\sum_F x_i^q+\sum_1^{k-1}y_i^q+\sum_q^\ell b_i+a\Big\|<1+\sum_1^{k-1}\varepsilon_i+\frac{\varepsilon}{1-\varepsilon}\ .$$

Choose $q'<q$ so that $\|x_i^q-x_i^{q'}\|,\|y_i^q-y_i^{q'}\|<\varepsilon$ for $i\le k-1$. Let $x_k=\sum_{q'}^{q-1}b_i$ and $y_k=\sum_q^\ell b_i$. We have $\big|\,\|x_k\|-1\big|,\big|\,\|y_k\|-1\big|<\varepsilon$ and for $F\subseteq\{1,\dots,k-1\}$

$$1)\qquad \Big\|\sum_F x_i^q+\sum_1^{k-1}y_i^q+y_k+a\Big\|<1+\sum_1^{k-1}\varepsilon_i+\frac{\varepsilon}{1-\varepsilon}\ ,$$

$$2)\qquad \Big\|\sum_F x_i^{q'}+\sum_1^{k-1}y_i^{q'}+x_k+y_k+a\Big\|<1+\sum_1^{k-1}\varepsilon_i+\frac{\varepsilon}{1-\varepsilon}\ .$$

It follows that if we set $x_i=x_i^q$, $y_i=y_i^q$ for $i\le k-1$ then for $F\subseteq\{1,\dots,k\}$

$$3)\qquad \Big\|\sum_F x_i+\sum_1^k y_i+a\Big\|\le 1+\sum_{i=1}^{k-1}\varepsilon_i+\frac{\varepsilon}{1-\varepsilon}+(2k-2)\varepsilon\ .$$

Thus Step 2 follows by our choice of ε.

Applying Step 2 to an arbitrary $a_k > e_{m_k}$, $\|a_k\| = 1$ we obtain that for all k there exists $x_i^k < y_i^k \in \langle e_j \rangle_{m_{k-1}+1}^{m_k}$ with $\big| \|x_i^k\| - 1 \big|, \big| \|y_i^k\| - 1 \big| < \varepsilon_i$ for $i \le k$ and also for $F \subseteq \{1, \dots, k\}$,

$$\Big\| \sum_F x_i^k + \sum_1^k y_i^k + a_k \Big\| < 2 \ .$$

It follows that $\| \sum_F x_i^k \| < 4$.

Choose $(k_j) \in [\mathbb{N}]$ so that for all i, $\lim_{j \to \infty} x_i^{k_j} \equiv x_i$ exists. We have for all $F \in [\mathbb{N}]^{<\omega}$,

$$\Big\| \sum_F x_i \Big\| \le 4 \quad \text{and} \quad \big| \|x_i\| - 1 \big| < 1 + \varepsilon_i \ .$$

Thus (x_i) is equivalent to the unit vector basis of c_0. $\square$

We end this section by presenting the

Proof of corollary to Theorem A.

If the corollary is false then all such s_i's satisfy $\|s_1 + s_2\| = 1$ or 2 and by Theorem A both occur. Thus the statement of the corollary can be obtained from the following observation

Remark 3.2. Let (e_i) be a basis for X and let $I(X) = \{r :$ there exists a normalized block basis of (e_i) having spreading model (s_i) with $\|s_1 + s_2\| = r\}$. Then $I(X)$ is a subinterval of $[1, 2]$.

Indeed, let $r_1 < r_2$ be in $I(X)$ and let (y_n) and (z_n) be normalized block bases of (e_i) with spreading models (s_i) and (t_i), respectively, satisfying $\|s_1 + s_2\| = r_1$ and $\|t_1 + t_2\| = r_2$. We may assume that $(y_1, z_1, y_2, z_2, \dots)$ is a block basis of (e_i). Furthermore, by a diagonal argument, we may assume that $(\alpha y_n + \beta z_n)_{n=1}^\infty$ has a spreading model $(s_n^{\alpha,\beta})_{n=1}^\infty$ for all $\alpha, \beta \in \mathbb{R}$ (not both 0). Now $s_n^{1,0} = s_n$ and $s_n^{0,1} = t_n$. There exists a continuous curve $\gamma : [0,1] \to \mathbb{R}^2$, $\gamma(t) = (\alpha(t), \beta(t))$ so that $\|s^{\alpha(t),\beta(t)}\| = 1$ for all $t \in [0,1]$ and $\gamma(0) = (1,0)$, $\gamma(1) = (0,1)$. We thus obtain by continuity that for all $r \in (r_1, r_2)$ there exists t with

$$\|s_1^{\alpha(t),\beta(t)} + s_2^{\alpha(t),\beta(t)}\| = r \ . \qquad \square$$

In the next section, Proposition 4.4, we shall see that $I(X)$ need not be closed.

§4. Theorem B and spreading models of T.

If (e_i) is a basic sequence and $x, y \in \langle e_i \rangle$ we say that x *equals* y *in distribution* ($x \overset{\mathcal{D}}{=} y$) if there exist $(n(i)), (m(i)) \in [\mathbb{N}]$ so that $\sum x(i) e_{n(i)} = \sum y(i) e_{m(i)}$. The *distance in distribution* between x and y is defined as

$$d(x,y) = \inf\left\{ \|\tilde{x} - \tilde{y}\| : \tilde{x} \overset{\mathcal{D}}{=} x \text{ and } \tilde{y} \overset{\mathcal{D}}{=} y \right\} .$$

Note that d defines a pseudometric on $< e_i >$. For $E \subseteq \mathbb{N}$ we set $Ex = \sum_{i\in E} x(i) e_i$.

In order to prove Theorem B we first prove

Proposition 4.1. *Let (e_i) be a normalized basic sequence having spreading model (s_i) which is K-equivalent to the unit vector basis of ℓ_1. Assume that there exists a normalized block basis (x_i) of (s_i) which satisfies*

1) $\lim_i \|x_i\|_{\ell_1} = K$ *and*
2) (x_i) *is Cauchy with respect to the distance in distribution*

(i.e., for all $\varepsilon > 0$ there exists n_0 with $d(x_n, x_m) < \varepsilon$ if $n, m \geq n_0$). Then there exists a block basis (y_i) of (e_i) having spreading model 1-equivalent to the unit vector basis of ℓ_1.

Proof. Let $x_i = \sum_{j\in A_i} a_j^i s_j$ for some choice of scalars and sets of integers $A_1 < A_2 < \cdots$.

Using 1), 2) and the fact that (s_i) is K-equivalent to the unit vector basis of ℓ_1 we have the following.

3) For all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$ there exists $F_n = F_n(\varepsilon) \subseteq A_n$ with $|F_n| \leq |A_{n_0}|$, $\|F_n x_n\|_{\ell_1} > K - \varepsilon$, $d(F_n x_n, x_{n_0}) < \varepsilon$ and $\sum_{i\in A_n \setminus F_n} |a_i^n| < \varepsilon$.

For $n \in \mathbb{N}$ set $y_n = \sum_{i\in A_n} a_i^n e_i$. Let $k \in \mathbb{N}$ and $(a_i)_1^k \subseteq \mathbb{R}$ with $\sum_1^k |a_i| = 1$. If $n_0 < n_1 < \cdots < n_k$ then from 3) we obtain

$$\left\| \sum_1^k a_i y_{n_i} \right\| \geq \left\| \sum_1^k a_i F_{n_i} y_{n_i} \right\| - \varepsilon .$$

Also

$$\Big|\bigcup_1^k \mathrm{supp}(F_{n_i}y_{n_i})\Big| \le k|A_{n_0}|.$$

Thus for all $\varepsilon > 0$,

$$\begin{aligned}\liminf_{\substack{n_1\to\infty\\ n_1<\cdots<n_k}} \Big\|\sum_1^k a_i y_{n_i}\Big\| &\ge \liminf_{\substack{n_1\to\infty\\ n_1<\cdots<n_k}} \Big\|\sum_1^k a_i F_{n_i}(\varepsilon)x_{n_i}\Big\| - \varepsilon\\ &\ge \liminf_{\substack{n_1\to\infty\\ n_1<n_2<\cdots<n_k}} \frac{1}{K}\Big\|\sum_1^k a_i F_{n_i}(\varepsilon)x_{n_i}\Big\|_{\ell_1} - \varepsilon \ge \frac{K-\varepsilon}{K} - \varepsilon\ ,\end{aligned}$$

where $\|\cdot\|_{\ell_1}$ refers to the ℓ_1-norm w.r.t. the coordinates (s_i). Since $\lim_n \|y_n\| = 1$ (e.g., use 3)) we obtain that (y_n) has a spreading model 1-equivalent to the unit vector basis of ℓ_1. □

Our argument was motivated by [J].

T (see e.g., [CS], [FJ]) is the completion of the linear space of finitely supported real valued sequences under the implicit norm

$$\|x\| = \|x\|_\infty \vee \sup\Big\{\tfrac{1}{2}\sum_{i=1}^n \|E_i x\| : n \in \mathbb{N}\ ,\ n \le E_1 < \cdots < E_n\Big\}\ .$$

Proof of Theorem B. We may assume that X has a basis (b_i) which is a block basis of (t_i), the unit vector basis for T. Let (e_i) be a normalized block basis of (b_i) where e_i is a $(1+\varepsilon_i) - \ell_1^{m_i}$ average for some sequences $m_i \uparrow \infty$ and $\varepsilon_i \downarrow 0$. Thus $e_i = (\sum_1^{m_i} \omega_j)/\|\sum_1^{m_i}\omega_j\|$ where $(\omega_j)_1^{m_i}$ a normalized block basis of (b_n) which is $(1+\varepsilon_i)$-equivalent to the unit vector basis of $\ell_1^{m_i}$. By passing to a subsequence we may assume that (e_i) has a spreading model (s_i).

Let $x \in \langle t_i\rangle$ be fixed with $x \le t_n$ and let $k \in \mathbb{N}$, $(a_i)_1^k \subseteq \mathbb{R}$. Then

$$\begin{aligned}&\overline{\lim_{\substack{n_1\to\infty\\ n_1<\cdots<n_k}}}\ \sup\Big\{\tfrac{1}{2}\sum_{j=1}^{\ell} \Big\|E_j\Big(x + \sum_1^k a_i e_{n_i}\Big)\Big\| : \ell \le E_1 < \cdots < E_\ell\ ,\ \ell \le n\Big\}\\ &\quad \le \|x\| + \tfrac{1}{2}\sum_1^k |a_i|\ .\end{aligned}$$

This follows from the fact that since e_{n_i} is a $(1+\varepsilon_{n_i}) - \ell_1^{m_{n_i}}$ average

$$\lim_{i\to\infty} \sup\left\{ \tfrac{1}{2} \sum_{j=1}^{\ell} \|E_j a_i e_{n_i}\| : \ell \le E_1 < \cdots < E_\ell \ , \ \ell \le n \right\}$$
$$= \tfrac{1}{2}|a_i| \quad \text{(see e.g., [OTW]).}$$

It follows that

$$\Big\| \sum_1^k a_i s_i \Big\| \le \sup_i \Big(|a_i| + \tfrac{1}{2} \sum_{i+1}^{k} |a_j| \Big) \ .$$

However since $\lim_i \|e_i\|_\infty = 0$ if k is fixed and $\varepsilon > 0$, then for i sufficiently large we have for some choice of $E_1 < \cdots < E_\ell$ that $1 = \|e_i\| \le \frac{1}{2}\sum_1^\ell \|E_j e_i\| + \varepsilon$ where $\ell \le m - k$, $m = \min E_1$. This yields that for all k, $(a_i)_1^k \subseteq \mathbb{R}$,

$$\Big\| \sum_1^k a_i s_i \Big\| = \max_i \Big(|a_i| + \tfrac{1}{2} \sum_{i=1}^{k} |a_j| \Big) \ . \tag{4}$$

All that remains is to show that Proposition 3.1 applies to (e_i) and (s_i). (s_i) is 2-equivalent to the unit vector basis of ℓ_1. Set

$$\begin{aligned} x_1 &= \tfrac{1}{2}s_1 + \tfrac{2}{3}s_2 \ , \\ x_2 &= \left(\tfrac{2}{3}\right)^2 s_3 + \left(\tfrac{2}{3}\right)^2 s_4 + \tfrac{2}{3}s_5 \ , \\ x_3 &= \left(\tfrac{2}{3}\right)^3 s_6 + \left(\tfrac{2}{3}\right)^3 s_7 + \left(\tfrac{2}{3}\right)^2 s_8 + \tfrac{2}{3}s_9 \ , \quad \text{etc.} \end{aligned}$$

In general x_n has the same distribution as

$$\left(\tfrac{2}{3}\right)^n s_1 + \left(\tfrac{2}{3}\right)^n s_2 + \left(\tfrac{2}{3}\right)^{n-1} s_3 + \cdots + \tfrac{2}{3}s_{n+1}$$

and (x_n) is a block basis of (s_i). It is easy to check by (4) that $\|x_n\| = 1$ and $\lim_n \|x_n\|_{\ell_1} = 2$. Also (x_n) is Cauchy in distribution since for $n < m$

$$d(x_n, x_m) \le \left(\tfrac{2}{3}\right)^n + \sum_{n+1}^{m} \left(\tfrac{2}{3}\right)^i + \left(\tfrac{2}{3}\right)^m \ . \qquad \square$$

Remark 4.2. The above argument yields the following. Let (x_i) be a normalized basic sequence having spreading model (e_i) equivalent to the unit vector basis of ℓ_1. Let

$$K \equiv \sup\left\{ \sum_1^n |a_i| : \Big\| \sum_1^n a_i e_i \Big\| = 1 \right\} \ .$$

Let $E_{\mathbb{Q}}$ be the completion of $\langle e_q : q \in \mathbb{Q}\rangle$ under $\|\sum_1^n a_i e_{q_i}\| = \|\sum_1^n a_i e_i\|$ if $q_1 < \cdots < q_n$. Suppose there exists $x \in E_{\mathbb{Q}}$ with $\|x\| = 1$ and $\|x\|_{\ell_1} = K$. Then there exists a normalized block basis (y_i) of (x_i) having spreading model 1-equivalent to the unit vector basis of ℓ_1. However such an x need not exist (consider $\|x\| = \|x\|_{c_0} + \|x\|_{\ell_1}$).

Remark 4.3. If $|\cdot|$ is any equivalent norm on T then for all $\varepsilon > 0$ there exists a spreading model of a normalized block basis of (e_i) which is $1+\varepsilon$-equivalent to the unit vector basis of ℓ_1. In fact one has [BL, p.43] more generally if (e_i) is a basic sequence with spreading model equivalent to the unit vector basis of ℓ_1, then for all $\varepsilon > 0$ there exists a spreading model of a normalized block basis of (e_i) which is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_1.

In [OS] it is proved that if X does not contain ℓ_1 then X can be renormed so that if (s_i) is a spreading model of a normalized sequence (x_n) then $\|s_1 + s_2\| < 2$ implies that (x_n) is not weakly null. Here we give an explicit renorming of T with this property. Since the arguments are quite technical and since the existence of such norm follows from the above cited result in [OS] we omit a proof.

Proposition 4.4. *Let $0 < q < 1/2$ and let $|\!|\!| \cdot |\!|\!|$ be defined on c_{00} by the implicit equation*

$$|\!|\!| x |\!|\!| = \|x\|_\infty \vee \sup\left\{ \frac{1}{2}\sum_{i=1}^n |\!|\!| E_i x |\!|\!| + q \max_{j\le n} |\!|\!| E_j x |\!|\!| : n \in \mathbb{N} \text{ and } n \le E_1 < \cdots < E_n \right\}.$$

Then $|\!|\!| \cdot |\!|\!|$ is an equivalent norm on T and if (s_n) is a spreading model of a normalized block basis of $(T, |\!|\!| \cdot |\!|\!|)$ then

$$|\!|\!| s_1 + s_2 |\!|\!| < 2 .$$

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