STRICTLY SINGULAR, NON-COMPACT OPERATORS EXIST ON THE SPACE OF GOWERS AND MAUREY

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Abstract We construct a strictly singular non-compact operator on Gowers' and Maurey's space GM.

1. Introduction

In 1993 W.T. Gowers and B. Maurey [GM] solved the famous "unconditional basic sequence problem" by constructing the first known example of a space that does not contain any unconditional basic sequence. In the present paper this space is denoted by GM. Furthermore it was shown in [GM] that the space GM is *hereditarily indecomposable* (HI), i.e. no infinite dimensional subspace of GM can be decomposed into a direct sum of two further infinite dimensional closed subspaces. As shown in [GM] every bounded operator on a complex HI space can be written as a sum of a multiple of the identity and a *strictly compact operator*. Recall that an operator T is strictly singular if no restriction of T to an infinite dimensional subspace is an isomorphism. Actually Lemma 22 of [GM] implies immediately that the real version of GM has also the property that every operator on a subspace of it is a strictly singular perturbation of a multiple of the identity. In [GM] it is asked whether or not every operator on GM can be written as a compact perturbation of a multiple of the identity. If the answer to this question were positive then by [AS] the space GM would be the first known example of an infinite dimensional Banach space such that every operator on it has a non-trivial invariant subspace i.e. GM would be a positive solution to the invariant subspace problem.

In 1999 W.T. Gowers showed that strictly singular non-compact operators can be defined on certain subspaces of GM, [G]. Our main result is

Theorem 1.1. *There exists a strictly singular non-compact operator* $T\colon\ GM \to GM$.

Since strictly singular non compact operators on an infinite dimensional banach space cannot be written as compact perturbations of a multiple of the identity, we give a negative answer to the question of W.T. Gowers and B. Maurey. Also, we show that the space of operators on GM contains a subspace isometric to ℓ_∞, the Banach space of all bounded sequences of scalars. Thus the set of operators on GM is non-separable in contrast to the separability of the set of compact perturbations of a multiple of the identity on a space with a basis.

Concerning the invariant subspace problem, note that if every strictly singular operator on GM has some compact power, then by [L], GM would still be a positive solution to the invariant subspace problem. However, we conjecture that there exists a strictly singular operator on GM none of whose powers is compact.

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The construction of the space GM is based on the space S, the first known example of an arbitrarily distortable space, constructed by the second named author in [S1].

The space S was used by E. Odell and the second author in order to show that the separable Hilbert space is arbitrarily distortable [OS]. The second named author proved in [S2] that the space S is complementably minimal. This means that for every infinite dimensional subspace of S there exists a further subspace which is isomorphic to S and complemented in S. Note that if a Banach space X with the Aproximation Property (AP) is complementably minimal then every non-trivial closed two-sided operator ideal $\mathcal{I}$ of X satisfies $\mathcal{K} \subseteq \mathcal{I} \subseteq \mathcal{S}$ where $\mathcal{K}$ is the ideal of compact operators and $\mathcal{S}$ is the ideal of strictly singular operators on X. Indeed, since $\mathcal{I}$ is closed and non-trivial, and X has the AP we have $\mathcal{K} \subseteq \mathcal{I}$. Assume that there exists $T \in \mathcal{I} \backslash \mathcal{S}$. Since T is not strictly singular, there exists an infinite dimensional subspace Y of X such that T restricted on Y is an isomorphism. Since X is complementably minimal there exists a subspace Z of $T(Y)$ such that Z is isomorphic to X and complemented in X. Let $j\colon\ X \to Z$ be an onto isomorphism and let $P\colon\ X \to Z$ be an onto projection. Finally note that ATB is the identity on X where $A = j^{-1}P$ and $B = (T|_Z)^{-1}j$ are operators on X. Therefore id $\in \mathcal{I}$ and $\mathcal{I}$ consists of all operators on X. Hence, for a complementably minimal space X having (AP), the lack of strictly singular non-compact operators on X is equivalent to X being *simple*. Recall that a Banach space X is simple if the only two-sided closed operator ideal on X is the ideal of compact operators. The only known simple spaces are the spaces ℓ_p $(1 \leq p < \infty)$ and c_0 [H] (see also [FGM] and [P] page 82). W.B. Johnson asked whether or not S is simple. We prove

Theorem 1.2. *There exists a strictly singular non-compact operator on S.*

Thus S is not a simple space. Also Theorem 1.1 implies that GM is not a simple space. We do not know how many two-sided closed operator ideals exist on S and on GM.

It will follow from our work that formaly the same operator T can be considered in both Theorems 1.1 and 1.2 (either as an operator on GM or as an operator on S). The operator T has the form $T = \Sigma x_i^* \otimes e_i$ where (x_i^*) is a seminormalized block sequence in S^* as well as in GM^* and (e_i) is the unit vector basis of S or GM respectively.

We assume all our Banach spaces being defined over $\mathbb{R}$ noting that the result can be easily transfered to the complex case.

We now recall the definition of S and will introduce first some basic notation. c_{00} is the vectorspace of sequences in $\mathbb{R}$ for which only finitely many coordinates are not zero. For $x \in c_{00}$ the *support* of x is the set $\{i \in \mathbb{N} : x_i \neq 0\}$ which we denote by $\mathrm{supp}(x)$. (e_i) is the usual unit basis in c_{00}, i.e. $e_i(j) = \delta_{ij}$, for $i, j \in \mathbb{N}$. If $x = \sum x_i e_i \in c_{00}$ and $E \subset \mathbb{N}$ we write $E(x) = \sum_{i \in E} e_i$. For two finite sets $E, F \subset \mathbb{N}$ we write $E < F$ if $\max E < \min F$ $(\max \emptyset = 0)$, and for $x, y \in c_{00}$ we write $x < y$ if $\mathrm{supp}(x) < \mathrm{supp}(y)$.

Let $\|\cdot\|_{\ell_p}$ denote the usual norm on ℓ_p if $1 \leq p \leq \infty$. Let f denote the function $f(x) = \log_2(x+1)$. S is the completion of $(c_{00}, \|\cdot\|_S)$ and $\|\cdot\|_S$ is the unique norm on c_{00} which satisfies the implicit formula:

$$\|x\|_S = \|x\|_{\ell_\infty} \vee \sup_{\substack{2 \leq n \in \mathbb{N} \\ E_1 < E_2 < \cdots < E_n}} \frac{1}{f(n)} \sum_{j=1}^{n} \|E_j x\|_S$$

The unique existence of such a norm is easy to show and it is also not hard to prove that (e_i) is a 1-unconditional and 1-subsymmetric basis of S (see [S1]).

For $\ell \in \mathbb{N}$, $\ell \geq 2$ we define the equivalent norm $\|\cdot\|_\ell$ on S by

$$\|x\|_\ell = \sup_{E_1 < \cdots < E_\ell} \frac{1}{f(\ell)} \sum_{j=1}^{\ell} \|E_j X\|_S.$$

The construction of the space GM will be recalled in Section 3.

2. Existence of strictly singular, non-compact operators on S

The main goal of this section is to prove Theorem 1.2. We start by stating a sufficient condition for an operator $T\colon\ S \to S$, of the form

$$T = \sum_i x_i^* \otimes e_i,$$

to be strictly singular but not compact.

Proposition 2.1. *Assume that $(x_i^*)_{i=1}^\infty$ is a seminormalized block sequence in S^* and that there is an increasing sequence $(C(\ell))_{\ell \in \mathbb{N}} \subset \mathbb{R}^+$, with $C(\ell) \nearrow \infty$, if $\ell \nearrow \infty$, for which the following condition holds.*

(1) *If $(z_i)_{i=1}^\infty$ is a block sequence in S, so that for each $i \in \mathbb{N}$, $x_i^*(z_i) = 1$ and $x_{i-1}^* < z_i < x_{i+1}^*$ (take $x_0^* = 0$), then for any $2 \leq \ell \in \mathbb{N}$ and $(\lambda_i)_{i=1}^\infty \in c_{00}$ we have that*

$$\left\| \sum_{i=1}^\infty \lambda_i e_i \right\|_\ell \leq \frac{1}{C(\ell)} \left\| \sum_{i=1}^\infty \lambda_i z_i \right\|_S.$$

Then the operator $T = \sum_{i=1}^\infty x_i^* \otimes e_i$, with $T(x) = \sum_{i=1}^\infty x_i^*(x) \cdot e_i$, for $x \in S$, is bounded, strictly singular, but not compact.

Proof. In order to see that $T\colon\ S \to S$ is bounded let $x \in c_{00}$. We can write x as $x = \sum_{i=1}^\infty \lambda_i z_i$, where (z_i) is a block sequence in S so that $x_i^*(z_i) = 1$ and $x_{i-1}^* < z_i < x_{i+1}^*$. Thus $Tx = \Sigma \lambda_i e_i$. If $\|Tx\|_S = \|Tx\|_{\ell_\infty}$, then

$$\|Tx\|_{\ell_\infty} = \max_{i \in \mathbb{N}} |\lambda_i| = \max_{i \in \mathbb{N}} \left| x_i^* \left(\sum \lambda_i z_j \right) \right| \leq \sup_{i \in \mathbb{N}} \|x_i^*\|_{S^*} \|x\|_S.$$

If $\|Tx\|_S = \|Tx\|_\ell$ for some $\ell \in \mathbb{N}$, $\ell \geq 2$, then it follows from our assumption (1) that

$$\|Tx\|_\ell = \left\| \sum_{i=1}^\infty \lambda_i e_i \right\|_\ell \leq \frac{1}{C(\ell)} \left\| \sum \lambda_i z_i \right\|_S \leq \frac{1}{C(2)} \|x\|_S.$$

In order to show that T is strictly singular we consider an arbitrary infinite dimensional subspace X of S, and let $\ell_0 \in \mathbb{N}$. X contains an element x for which there exists an $\ell \geq \ell_0$ so that $\|x\|_S = \|x\|_\ell$. Indeed, as it was shown in [S1], we find for any $N \in \mathbb{N}$ and $\varepsilon > 0$ a normalized block $(y_i)_{i=1}^N$ in X which is $(1+\varepsilon)$-equivalent to the ℓ_1^N-unit vector basis, in particular it follows that

$$\frac{1}{1+\varepsilon} \leq \left\| \frac{1}{N} \sum_{i=1}^N y_i \right\|_S.$$

On the other hand it was shown in [S1] that given $\ell_0 \in \mathbb{N}$ we can choose N big enough so that for any $m \in \mathbb{N}$, $m \leq \ell_0$, it follows that $\|\frac{1}{N}\sum_{i=1}^N y_i\|_m < (1+\delta(\varepsilon))/f(m)$, with $\delta(\varepsilon) \downarrow 0$, for $\varepsilon \downarrow 0$. Since clearly $\|\frac{1}{N}\sum_{i=1}^N y_i\|_{\ell_\infty} \leq \frac{1}{N}$ there exists an $\ell \geq \ell_0$ so that

$\|\frac{1}{N}\sum_{i=1}^N y_i\|_\ell = \|\frac{1}{N}\sum_{i=1}^N y_i\|_S$. Now assume that X is an infinite dimensional subspace of S on which T acts as an isomorphism. For any ℓ_0 we can choose a $y \in T(X)$, $y \in T(x)$, so that $\|y\| = \|y\|_\ell$, with $\ell \geq \ell_0$. As before we write x as $x = \sum_{i=1}^\infty \lambda_i z_i$ where (z_i) is a block sequence with $x_i^*(z_i) = 1$ and $x_{i-1}^* < z_i < x_{i+1}^*$ for $i \in \mathbb{N}$. Then $\|Tx\|_S = \|Tx\|_\ell = \|\sum_{i=1}^\infty \lambda_i e_i\|_\ell \leq \frac{1}{C(\ell)}\|\sum_{i=1}^\infty \lambda_i z_i\|_S \leq \frac{1}{C(\ell_0)}\|x\|_S$. Since $\ell_0 \in \mathbb{N}$ was arbitrary and since $C(\ell) \nearrow \infty$ for $\ell \nearrow \infty$ we arrive to a contradiction to the assumption that $T|_X$ in an isomorphism.

Finally, T is not compact since (x_i^*) is a seminormalized block sequence. □

Remark. Using the same proof, Proposition 2.1 can be generalized as follows: Assume $(x_i^*)_{i=1}^\infty \subset S^*$ and $(y_i)_{i=1}^\infty \subset S$ are seminormalized block sequences and $(C(\ell)) \subset \mathbb{R}_+$ so that $C(\ell) \nearrow \infty$ if $\ell \nearrow \infty$ and the following conditions holds:
If $(z_i)_{i=1}^\infty$ is a block sequence in S, so that for each $i \in \mathbb{N}$, $x_i^*(z_i) = 1$ and $x_{i-1}^* < z_i < x_{i+1}^*$, then for any $2 \leq \ell \in \mathbb{N}$ and $(\lambda_i)_{i=1}^\infty \in c_{00}$ we have that

$$\Big\|\sum \lambda_i y_i\Big\|_\ell \leq \frac{1}{C(\ell)}\Big\|\sum \lambda_i z_i\Big\|_S + \max|\lambda_i|.$$

Then the operator $T = \Sigma x_i^* \otimes y_i$ is bounded, strictly singular but not compact.
Remark. The rest of this section is devoted to the construction of a semi normalized block (x_n^*) in S^* and a sequence $C(\ell)_{\ell\in\mathbb{N}}$ satisfying the condition stated in Proposition 2.1. To do that we will have to overcome several technical difficulties. In a first reading one can proceed directly to Section 3. There we will show that by spreading out the coordinates of the sequence (x_i^*) and under some mild growth condition on $C(\ell)$ (which will be verified) the operator $\Sigma x_i^* \otimes y_i$ is well defined on GM, strictly singular and non-compact.

In order to define the sequence $(x_m^*) \subset S^*$ we will need the following notion of *finitely branching trees.* Consider the set $\cup_{k=0}^\infty \mathbb{N}^k$, the set of all finite sequences with values in $\mathbb{N}$, which will be partially ordered by extensions, i.e. if $\mu = (\mu_1, \ldots, \mu_m)$ and $\nu = (\nu_1, \ldots, \nu_n)$ are in $\cup_{k=0}^\infty \mathbb{N}^k$ we write $\mu \prec \nu$ if $m \leq n$ and $\mu_1 = \nu_1$, $\mu_2 = \nu_2, \ldots, \mu_m = \nu_m$. For $\mu = (\mu_1, \ldots, \mu_m) \in \cup_{k=0}^\infty \mathbb{N}^k$, we call m the *length of* μ and denote it by $|\mu|$.

A non-empty subset $\mathcal{T}$ of $\cup_{k=0}^\infty \mathbb{N}^k$ is called a *finitely branching tree*, or simply a tree (since we will not deal with other kinds of trees) if the following two conditions hold.

(2) If $\mu = (\mu_1, \ldots, \mu_m) \in \mathcal{T}$ and $0 \leq k \leq m$, then also $(\mu_1, \ldots, \mu_k)$ lies in $\mathcal{T}$.

Since $\mathcal{T}$ is non-empty, this implies in particular that $\emptyset \in \mathcal{T}$.

(3) If $\mu \in \mathcal{T}$, then either μ is maximal in $\mathcal{T}$, i.e. no extension of μ lies in $\mathcal{T}$, or there is a $k_\mu = k_\mu^{\mathcal{T}} \in \mathbb{N}$ so that $\{(\mu, i):\ i \in \{1, 2, \ldots, k_\mu\}\} = \{\nu \in \mathcal{T}:\ \mu \prec \nu, |\nu| = |\mu| + 1\}$.

We call the elements of trees *nodes*. If $\mathcal{T}$ is a tree and $\mu = (\mu_1, \ldots, \mu_{|\mu|}) \in \mathcal{T}\backslash\{\emptyset\}$ we call $(\mu_1, \ldots, \mu_{|\mu|-1})$ the *immediate predecessor* of μ. If $\mu \in \mathcal{T}$ is not maximal in $\mathcal{T}$ we call the nodes (μ, i), $i \leq k_\mu^{\mathcal{T}}$, *immediate successors* of μ in $\mathcal{T}$. Since we will always deal with only one tree $\mathcal{T}$ at a time we denote the numbers of successors of an non-maximal $\mu \in \mathcal{T}$ simply by k_μ. We say that a tree $\mathcal{T}$ is of length ℓ, $\ell \in \mathbb{N}_0$, if $\mathcal{T} \subset \cup_{k=0}^\ell \mathbb{N}^k$ and all maximal nodes in $\mathcal{T}$ have length ℓ. A tree is said to have infinite length if it does not contain maximal nodes. On a tree $\mathcal{T}$ having finite or infinite length we also introduce the *lexicographic order* which we denote by $\prec_{\text{lex}}$, i.e. we well-order $\mathcal{T}$ into: $\emptyset, (1), (2), \ldots, (k_\emptyset)$, $(1,1), \ldots, (1, k_{(1)})$, $(2,1), \ldots, (2, k_{(2)})$, $(3,1), \ldots$. To a given tree $\mathcal{T}$ having finite length we want to define *associated vectors* in S and S^*. These vectors are defined up to *equality in distribution*. For $x, y \in c_{00}$ we say that x and y have the same distribution and write $x =_{\text{dist}} y$ if for some

$k \in \mathbb{N}$, $(\alpha_i)_{i=1}^k \subset \mathbb{R}$, and $m_1 < m_2 < \ldots < m_k$ and $n_1 < n_2 < \cdots < n_k$ in $\mathbb{N}$ it follows that $x = \sum_{i=1}^k \alpha_i e_{m_i}$ and $y = \sum_{i=1}^k \alpha_i e_{n_i}$.

Definition. Let $\mathcal{T}$ be a tree of length $\ell \in \mathbb{N}_0$. For $k \le \ell$ and each $\mu = (\mu_1, \ldots, \mu_k) \in \mathcal{T}$ we define the numbers

$$\alpha(\mu) = \prod_{i=0}^{k-1} \frac{f(k_{(\mu_1,\ldots,\mu_i)})}{k_{(\mu_1,\ldots,\mu_i)}} \text{ and } \beta(\mu) = \prod_{i=0}^{k-1} \frac{1}{f(k_{(\mu_1,\ldots,\mu_i)})} \tag{4}$$

(if $k=0$, then $(\mu_1, \ldots, \mu_0) = \emptyset$ and $\alpha(\emptyset) = \beta(\emptyset) = 1$). We say that $x \in S$ is *associated to* $\mathcal{T}$ if

$$x = \sum_{\mu \in \mathcal{T}, |\mu| = \ell} \alpha(\mu) \cdot e_{n(\mu)} \tag{5}$$

and we say that $x^* \in S^*$ is *associated to* $\mathcal{T}$ if

$$x^* = \sum_{\mu \in \mathcal{T}, |\mu| = \ell} \beta(\mu) e^*_{n(\mu)} \tag{6}$$

where $(n(\mu))_{\mu \in \mathcal{T}} \in \mathbb{N}$ has the property that $n(\mu) < n(\nu)$ if $\mu \prec_{\text{lex}} \nu$.

Remark. In the above notation it is easy to see that $\|x\|_S \ge 1$, $\|x^*\|_{S^*} \le 1$ and that $x^*(x) = 1$ (if the $n(\mu)$'s coincide).

Remark. There is also a recursive way to introduce the vectors x and x^* which are associated to a tree of length ℓ. If $\ell = 0$, i.e. $\mathcal{T} = \{\emptyset\}$, the associated vectors in S and S^* are simply the elements of the unit vector basis. Assume that for an $\ell \ge 0$, we have defined the vectors associated to trees of length ℓ in S and S^* respectively. Let $\mathcal{T}$ be a tree of length $\ell + 1$. For $i \le k_\emptyset$, the number of immediate successors of $\emptyset$ in $\mathcal{T}$, we let $\mathcal{T}_i = \big\{\mu \in \cup_{j=0}^{\ell} \mathbb{N}^j \mid (i, \mu) \in \mathcal{T}\big\}$. Then $\mathcal{T}_i$ is a tree of length ℓ and we choose vectors $x_i \in S$ and $x_i^* \in S$ associated to $\mathcal{T}_i$. Furthermore we require $(x_i)_{i=1}^{k_\emptyset}$ and $(x_i^*)_{i=1}^{k_\emptyset}$ to be blocks. Then the vectors associated to $\mathcal{T}$ are

$$x = \frac{f(k_\emptyset)}{k_\emptyset} \sum_{i=1}^{k_\emptyset} x_i \quad \text{and} \quad x^* = \frac{1}{f(k_\emptyset)} \sum_{i=1}^{k_\emptyset} x_i^*. \tag{7}$$

More generally if $0 \le k \le \ell$, and if $\mathcal{T}$ is a tree of length ℓ, then every $x \in S$, respectively $x^* \in S$, associated to $\mathcal{T}$ can be written as

$$x = \sum_{\mu \in \mathcal{T}, |\mu| = k} \alpha(\mu) x_\mu, \quad \text{and} \quad x^* = \sum_{\mu \in \mathcal{T}, |\mu| = k} \beta(\mu) x^*_\mu, \tag{8}$$

where x_μ and x^*_μ are associated to $\mathcal{T}_\mu := \big\{\nu \in \cup_{j=0}^{\ell-k} \mathbb{N}^j \mid (\mu, \nu) \in \mathcal{T}\big\}$ and (x_μ) and (x^*_μ) are blocks with respect to the lexicographic order.

Every tree $\mathcal{T}$ is determined, by the numbers k_μ, for $\mu \in \mathcal{T}$ being non-maximal. In order to choose the wanted block sequence $(x_n^*) \subset S^*$ which satisfies the requirements of Proposition 2.1 we will first choose a "lacunary enough" subset $K \subset \mathbb{N}$. Secondly, we will choose an infinite tree $\mathcal{T}$, with $(k_\mu)_{\mu \in \mathcal{T}} \subset K$ and such that (k_μ) increases with respect to the lexicographic order. Then we will choose a sequence $(L_n)_{n=1}^\infty \subset \mathbb{N}$, which increases "fast enough" to ∞. Finally we let $(x_n^*) \subset S^*$ be a block sequence for which x_n^*, $n \in \mathbb{N}$, is associated to $\mathcal{T}_{L_n} = \{\mu \in \mathcal{T} \mid |\mu| \le L_n\}$. Let us formulate these conditions precisely.

Let $(\varepsilon_i)_{i=1}^\infty$ be a decreasing sequence of positive numbers so that $\sum_{i=1}^\infty \varepsilon_i < \infty$. Assume that $(k_i)_{i=1}^\infty \subset \mathbb{N}$ satisfies the following growth conditions

$$\frac{2}{3} f\left(\frac{3}{4} f(k_1)\right) \geq 1, \text{ and } \ln k_1 \geq 3 \tag{9}$$

$$\text{for} \quad r > k_1 \quad \text{and} \quad a > 1, \text{ then } f(r^a) \geq \frac{3}{4} a f(r). \tag{10}$$

Note that always $f(r^a) \leq a f(r)$, and that we can assume (10) for a large enough k_1 since $f(\cdot)$ is asymptotically logarithmic. For $j \in \mathbb{N}$ we require the following inequalities

$$\left(\frac{6}{3f(k_j)} \prod_{s=1}^{j-1} \frac{k_s}{f(k_s)}\right)^{1/2} \sum_{\ell=0}^\infty \left(\frac{8}{3}k_1\right)^{-\ell/2} + 2\frac{\log_2 f(k_j)}{f(k_j)^{1/2}} \left(\prod_{s=1}^{j-1} \frac{k_s}{f(k_s)}\right)^{1/2} < \varepsilon_j. \tag{11}$$

For $r > 1$ we define

$$G(r) := \max_{i \in \mathbb{N}} \frac{f(r) f(k_i)}{f(r k_i)} \prod_{j=1}^{i-1} \frac{f(k_j)}{k_j} \tag{12}$$

and establish several properties. First note that above maximum is welldefined since for fixed $r > 0$ $\frac{f(r)f(k_i)}{f(rk_i)} \prod_{j=1}^{i-1} \frac{f(k_j)}{k_j}$ converges to zero if $i \nearrow \infty$.

Proposition 2.2.

a) *$G(r)$ is increasing in $r > 1$.*

b) *If $(k_{i_j})_{j=1}^\infty$ is a subsequence of $(k_i)_{i=1}^\infty$ and we define*

$$\overline{G}(r) = \max_{j \in \mathbb{N}} \frac{f(r) f(k_{i_j})}{f(r k_{i_j})} \prod_{s=1}^{j-1} \frac{f(k_{i_s})}{k_{i_s}}$$

(i.e. we replace (k_i) in the definition of G by (k_{i_j})), then $\overline{G}(r) \geq G(r)$.

Proof. For (a) we have to show that the function $[1, \infty) \ni x \mapsto \frac{f(x)}{f(ax)}$, $a \geq k_1$ is increasing. By taking derivatives we need to show that

$$\frac{\ln(1+ax)}{1+x} - \frac{a\ln(1+x)}{1+ax} \geq 0$$

.

An easy computation shows that:

$$\frac{\ln(1+ax)}{1+x} - \frac{a\ln(1+x)}{1+ax} \geq \frac{\ln(ax)}{1+x} - \frac{a\ln(x)+a}{1+ax} \geq \frac{ax(\ln(a)-3)}{(1+x)(1+ax)} \geq 0 \text{ (by (9))}.$$

In order to show (b) let $r > 1$ and choose $i \in \mathbb{N}$ such that

$$G(r) = \frac{f(r)f(k_i)}{f(rk_i)} \prod_{s=1}^{i-1} \frac{f(k_s)}{k_s}.$$

Secondly choose an $s \geq 1$ so that $k_{i_{s-1}} \leq k_{i-1} < k_i \leq k_{i_s}$ ($k_0 = k_{i_0} = 1$). Then

$$\overline{G}(r) \geq \frac{f(r)f(k_{i_s})}{f(rk_{i_s})} \prod_{t=1}^{s-1} \frac{f(k_{i_t})}{k_{i_t}} \geq \frac{f(r)f(k_{i_s})}{f(rk_{i_s})} \prod_{t=1}^{i-1} \frac{f(k_t)}{k_t} \geq \frac{f(r)f(k_i)}{f(rk_i)} \prod_{t=1}^{i-1} \frac{f(k_t)}{k_t} = G(r).$$

□

Lemma 2.3. *For $i\in\mathbb{N}$ choose $m_i\in\mathbb{R}^+$ such that $f(m_ik_i)=k_i$, i.e. $m_i=\frac{2^{k_i-1}}{k_i}$. Assume that $r\in[m_{i-1},m_i)$, for some $i\ge 2$ and define the sequence $(r_\ell)_{\ell=0}^\infty$ inductively by $r_0=r$ and $r_{\ell+1}=r_\ell^{f(r_\ell)}$. Then*

$$\sum_{\ell=0}^{\infty}\frac{1}{\sqrt{G(r_\ell)}}\le\sum_{j=i}^{\infty}(\varepsilon_j+\varepsilon_{j-1}). \tag{13}$$

The reader interested in the technical details of the proof is referred to the end of this section.

In addition to the growth conditions on $(k_i)_{i=1}^\infty$ stated in (9), (10), and (11) we will need a further property. Recall from [S1] that for given $\ell\in\mathbb{N}$ and $\varepsilon>0$ one can choose $n_1<n_2<\ldots<n_\ell$ fast enough increasing so that a block $(y_i)_{i=1}^\ell$, with $y_i=_{\rm dist}\frac{f(n_i)}{n_i}\sum_{j=1}^{n_i}e_j$, for $i=1,\ldots,\ell$, is $(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^\ell$ in S. In particular this implies that $\left\|\sum_{i=1}^\ell y_i\right\|_S\le\frac{(1+\varepsilon)\ell}{f(\ell)}$. Therefore we can and will require the following additional property on the sequence $(k_i)_{i=1}^\infty$.

(14) If $\mathcal{T}$ is a tree of infinite length with $\{k_\mu\colon\mu\in\mathcal{T}\}\subset(k_i)_{i=1}^\infty$ and $k_\mu<k_\nu$, if $\mu\prec_{\rm lex}\nu$, then for $\ell\in\mathbb{N}$ and $x\in S$ associated to $\{\mu\in\mathcal{T}:\ |\mu|\le\ell\}$ it follows that $\|x\|\le 2$.

Note that (14) implies that for a tree $\mathcal{T}$ as in (14), and $\ell\in\mathbb{N}$, any $x^*\in S^*$ associated to $\{\mu\in\mathcal{T},|\mu|\le\ell\}$ is at least of norm $\frac12$. Assume now that $(k_i)_{i=1}^\infty$ satisfies (9) through (11) and (14), and let $G(r)$, $r>1$, be defined as in (12). We choose a sequence $(L_n)_{n=1}^\infty\subset\mathbb{N}$ which has the following property (15).

(15) For $r>1$, the number $L_{\lfloor f(f(r))\rfloor}$ is big enough to ensure that

$$G(r)=\max_{i\le L_{\lfloor f(f(r))\rfloor}}\frac{f(k_i)f(r)}{f(k_ir)}\prod_{j=1}^{i-1}\frac{f(k_j)}{k_j}.$$

Note that (15) passes through to any subsequence $(k_{i_s})_{s=1}^\infty$ of (k_i), i.e.

$$\overline{G}(r):=\max_{s\in\mathbb{N}}\frac{f(k_{i_s})f(r)}{f(k_{i_s}r)}\prod_{t=1}^{s-1}\frac{f(k_{i_t})}{k_{i_t}}=\max_{s\le L_{\lfloor f(f(r))\rfloor}}\frac{f(k_{i_s})f(r)}{f(k_{i_s}r)}\prod_{t=1}^{s-1}\frac{f(k_{i_t})}{k_{i_t}}.$$

Finally we choose an infinite tree $\mathcal{T}$ so that $\{k_\mu:\ \mu\in\mathcal{T}\}\subset(k_i)_{i=1}^\infty$ and so that $k_\mu<k_\nu$ if $\mu\prec_{\rm lex}\nu$. We let $(x_n^*)_{n=1}^\infty$ be a block sequence in S^*, with x_n^* being associated to the tree $\{\mu\in\mathcal{T}:\ |\mu|\le L_n+1\}$.

Lemma 2.4. *Assume that $(z_n)_{n=1}^\infty\subset S$ has the property that $x_n^*(z_n)=1$, and $x_{n-1}^*<z_i<x_{n+1}^*$. If $m\in\mathbb{N}$ and $I\subset\mathbb{N}$ are such that $f(f(m))\le\min I$ and $\#I\le m$, then*

$$\Big\|\sum_{i\in I}\lambda_iz_i\Big\|_S\ge\frac{G(m)}{f(m)}\sum_{i\in I}|\lambda_i|,\ \textit{whenever } (\lambda_i)_{i\in I}\subset\mathbb{R}.$$

Proof. Using (8) we can write for any n and $\ell=0,1,\ldots,L_n+1$ x_n^* as

$$x_n^*=\sum_{\substack{\mu\in\mathcal{T}\\|\mu|=\ell}}\beta(\mu)x_{n,\mu}^*$$

where $\beta(\mu) = 1/\prod_{i=1}^{|\mu|-1} f(k_{(\mu_1,\dots,\mu_i)})$ and $x^*_{n,\mu}$ is associated to $\{\nu \in \cup_{j=0}^{L_n+1-\ell}\mathbb{N}^j \mid (\mu,\nu) \in \mathcal{T}\}$, for $\mu \in \mathcal{T}$, with $|\mu| = \ell$, and where $(x^*_{n,\mu})_{\mu\in\mathcal{T},|\mu|=\ell}$ is a block with respect to $\prec_{\text{lex}}$. Assume $m \in \mathbb{N}$, $I \subset \mathbb{N}$, with $f(f(m)) \le \min I$ and $\#I \le m$, and let $(\lambda_i)_{i\in I} \subset \mathbb{R}$. Without loss of generality we can assume that $\lambda_i \ge 0$, for $i \in I$.

By recursion we will choose for each $\ell = 0, 1, \dots, L_{\lfloor f(f(m))\rfloor}$ a node $\nu_\ell \in \mathcal{T}$, with $|\nu_\ell| = \ell$ and $\nu_0 \prec \nu_1 \prec \nu_2 \prec \dots \prec \nu_\ell$, so that

$$\sum_{i\in I} \lambda_i x^*_{i,\nu_\ell}(z_i) \ge \prod_{s=1}^{\ell-1} \frac{f(k_{\nu_s})}{k_{\nu_s}} \sum_{i\in I} \lambda_i. \tag{16}$$

For $\ell = 0, \mu_0 = \emptyset$, and since $x^*_{n,\emptyset} = x^*_n$, the claim follows from the assumptions on (z_n). Assume $\nu_0 \prec \nu_1 \prec \dots \prec \nu_\ell$, with $\ell < L_{\lfloor f(f(m))\rfloor}$, have been chosen. Then it follows from (8) that

$$x^*_{n,\nu_\ell} = \frac{1}{f(k_{\nu_\ell})} \sum_{j=1}^{k_{\nu_\ell}} x^*_{i,(\nu_\ell,j)}$$

and by the induction hypothesis that

$$\prod_{s=0}^{\ell-1} \frac{f(k_{\nu_s})}{k_{\nu_s}} \sum_{i\in I} \lambda_i \le \sum_{i\in I} \lambda_i x^*_{i,\nu_\ell}(z_i) = \frac{1}{f(k_{\nu_\ell})} \sum_{i\in I} \sum_{j=1}^{k_{\nu_\ell}} \lambda_i x^*_{i,(\nu_\ell,j)}(z_i).$$

Thus there exists a $j \le k_{\nu_\ell}$ so that for $\nu_{\ell+1} := (\nu_\ell, j)$ it follows that

$$\sum_{i\in I} \lambda_i x^*_{i,\nu_{\ell+1}}(z_i) \ge \prod_{s=0}^{\ell} \frac{f(k_{\nu_s})}{k_{\nu_s}} \sum_{i\in I} \lambda_i.$$

This finishes the proof of the claim.

For $\ell = 0, 1, \dots, L_{\lfloor f(f(m))\rfloor}$ let

$$y^*_\ell = \frac{1}{f(mk_{\nu_\ell})} \sum_{i\in I} \sum_{j=1}^{k_{\nu_\ell}} x^*_{i,(\nu_\ell,j)} = \frac{f(k_{\nu_\ell})}{f(mk_{\nu_\ell})} \sum_{i\in I} x^*_{i,\nu_\ell}$$

(recall that x^*_i, with $i \in I$, is the associated vector to a tree of length at least $L_{\lfloor f(f(m))\rfloor} + 1$). Since $\#I \le m$ it follows that on the $\|y^*_\ell\|_S \le 1$, and, then, that

$$\begin{aligned}\Big\|\sum_{i\in I} \lambda_i z_i\Big\|_S &\ge \max_{\ell \le L_{\lfloor f(f(m))\rfloor}} y^*_\ell\Big(\sum_{i\in I} \lambda_i z_i\Big) = \max_{\ell \le L_{\lfloor f(f(m))\rfloor}} \frac{f(k_{\nu_\ell})}{f(mk_{\nu_\ell})} \sum_{i\in I} \lambda_i x^*_{i,\nu_\ell}(z_i)\\ &\ge \max_{\ell \le L_{\lfloor f(f(m))\rfloor}} \frac{f(k_{\nu_\ell})}{f(mk_{\nu_\ell})} \prod_{s=1}^{\ell-1} \frac{f(k_{\nu_s})}{k_{\nu_s}} \sum_{i\in I} \lambda_i = \frac{G(m)}{f(m)} \sum_{i\in I} \lambda_i.\end{aligned}$$

In the last of above inequalities we used (15), the remark after (15), and Proposition 2.1. $\square$

Before we can prove that our chosen sequence (x^*_n) satisfies the properties stated in Proposition 2.1 we will need another argument. For $r \ge 2$ and $x \in S$ define

$$|||x|||_r = \sup_{r \le \ell < \infty} \|x\|_\ell = \sup_{\ell \ge r, E_1 < E_2 < \dots < E_\ell} \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i x\|_S.$$

Note that $|||\cdot|||_r$ is an equivalent norm on S. The next lemma makes the following qualitative statement precise:

If $x \in c_{00}$ and if $r \geq 2$ is "big enough", if $\ell \geq r$ and $E_1 < E_2 < \cdots < E_\ell$ are such that

$$|||x|||_r = \|x\|_\ell = \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i x\|_S$$

then "for most of the" $i \in \{1, \ldots, \ell\}$ it follows that either $\|E_i x\|_S = \|E_i x\|_{\ell_\infty}$ or $\|E_i x\|_S = \|E_i x\|_{n_i}$ with n_i being "much bigger than r".

Lemma 2.5. *There is a $d > 1$, so that for all r with $f(r) > d^2$ and all $x = \sum x_j e_j \in c_{00}$, it follows that*

$$|||x|||_r = \|x\|_\ell \leq \frac{1}{1 - \frac{d}{\sqrt{f(r)}}} \frac{1}{f(\ell)} \left(\sum_{j \in J} |x_j| + \sum_{i=1}^{\ell - \#J} \|E_i x\|_{n_i} \right),$$

where $\ell \in \mathbb{N}$, $\ell \geq r$, $J \subset \operatorname{supp}(x)$, $\#J \leq \ell$, $E_1 < E_2 < \cdots < E_{\ell-\#J}$, $E_i \subset \mathbb{N}$ and $E_i \cap J = \emptyset$, and $n_1, n_2, \ldots, n_{\ell-\#J} \in \mathbb{N}$, $n_j > r^{f(r)}$ for $j = 1, 2, \ldots, \ell - \#J$.

Lemma 2.5 is almost identical to Lemma 5 in [S2]; for the sake of being self-contained we include a proof at the end of this section. Now we are ready for the last step of proving Theorem 1.2, by showing that (x_n^*) as chosen before satisfies Proposition 2.1.

Lemma 2.6. *There is a constant $c > 1$ and an $m_0 \in \mathbb{N}$, so that for any block sequence $(z_n) \subset S$, with $x_n^*(z_n) = 1$ and $x_{i-1}^* < z_i < x_{i+1}^*$, for $n \in \mathbb{N}$, and any $m \geq m_0$ it follows that*

$$\left\| \sum_{i=1}^{\infty} \lambda_i e_i \right\|_m \leq \frac{c}{\sqrt{G(m)}} \left\| \sum_{i=1}^{\infty} \lambda_i z_i \right\|_S$$

whenever $(\lambda_i) \in c_{00}$.

Choosing now $C(\ell) = \sqrt{G(\ell)}/c$ for $\ell \geq \min\{m \geq m_0 \colon \sqrt{G(m)} \geq c\}$, and $C(\ell) = 1$ for $\ell < \min\{m \geq m_0 : \sqrt{G(m)} \geq c\}$ we note that the assumptions of Proposition 2.1 are fulfilled. For the case $\ell < \min\{m \geq m_0 : \sqrt{G(m)} \geq c\}$ we are simply using the fact that every blockbasis in S whose elements are of norm at least 1 dominates (e_i) [S2].

Remark. It will be important for the arguments in Section 3 that for any $r > 1$ the series $\sum 1/C(r_\ell)$ converges, where $r_0 = r$ and, inductively, $r_{\ell+1} = r_\ell^{f(r_\ell)}$ (Lemma 2.3).

Proof of Lemma 2.6. We choose m_0 so that $f(m_0) \geq 2d^2$, where d is chosen as in Lemma 2.5, and so that $m_0 > k_1$, the first element of the previously chosen sequence (k_n). For $r \in \mathbb{R}_+$ with $f(r) \geq 2d^2$ we consider the sequence $(r_\ell)_{\ell=0}^\infty$, with $r_0 = r$ and $r_{\ell+1} = r_\ell^{f(r_\ell)}$ and observe that the two series $\sum_{\ell=0}^\infty \frac{d}{\sqrt{f(r_\ell)} - d}$ and $\sum_{\ell=0}^\infty \frac{f(f(r_\ell))\sqrt{G(r_\ell)}}{f(r_\ell)}$ are finite (note that $\sqrt{G(r_\ell)} \leq \sqrt{f(r_\ell)}$ and that $\frac{f(f(r_\ell))}{\sqrt{f(r_\ell)}}$ is summable). By Lemma 2.3 also the series $\sum_{\ell=0}^\infty \frac{1}{\sqrt{G(r_\ell)}}$ is finite. Therefore

$$c(r) := \prod_{\ell=0}^{\infty} \frac{1}{1 - \frac{d}{\sqrt{f(r_\ell)}}} \left[1 + \frac{f(f(r_\ell))\sqrt{G(r_\ell)}}{f(r_\ell)} + \frac{1}{\sqrt{G(r_\ell)} - 1} \right]$$

is uniformly bounded on $[m_0,\infty)$. By induction on $\#I$ we will show that

$$\Big\|\sum_{i\in I}\lambda_i e_i\Big\|_m \le \frac{c(m)}{\sqrt{G(m)}}\Big\|\sum_{i\in I}\lambda_i z_i\Big\|_S \text{ for all } m\ge m_0. \tag{17}$$

This would imply the statement of the Lemma if we choose $c=\sup_{m\ge m_0} c(m)$.

If $\#I=1$ the claim is trivial (note that $\|e_i\|_m = 1/f(m)\ge 1/\sqrt{G(m)}$).

Assume that for some $k\in\mathbb{N}$ and all $I\subset\mathbb{N}$, $\#I\le k$, (17) is true and assume $\#I=k+1$, $m\ge m_0$, and $(\lambda_i)_{i\in I}\subset\mathbb{R}$. By Lemma 2.5 we can choose $\ell\ge m$, $J\subset I$, $\#J\le\ell$, $E_1<E_2<\cdots<E_{\ell-\#J}$, $E_i\cap J=\emptyset$, and $n_1,n_2,\dots,n_{\ell-\#J}\in(\lfloor m^{f(m)}\rfloor,\infty)\cap\mathbb{N}$ so that (put $x=\sum_{i\in I}\lambda_i e_i$ and $z=\sum_{i\in I}\lambda_i z_i$)

$$\begin{aligned}
\Big\|\sum_{i\in I}\lambda_i e_i\Big\|_m &\le \Big|\Big\|\sum_{i\in I}\lambda_i e_i\Big\|\Big|_m = \Big\|\sum_{i\in I}\lambda_i e_i\Big\|_\ell \\
&\le \frac{1}{1-\frac{d}{\sqrt{f(m)}}}\frac{1}{f(\ell)}\Big[\sum_{j\in J}|\lambda_j| + \sum_{i=1}^{\ell-\#J}\|E_i x\|_{n_i}\Big] \\
&\le \frac{1}{1-\frac{d}{\sqrt{f(m)}}}\frac{1}{f(\ell)}\Big[\sum_{\substack{j\in J\\ j\le f(f(\ell))}}|\lambda_i| + \sum_{\substack{j\in J\\ j> f(f(\ell))}}|\lambda_i| + \sum_{i=1}^{\ell-\#J}\|E_i x\|_{n_i}\Big] \\
&\le \frac{1}{1-\frac{d}{\sqrt{f(m)}}}\Big[\frac{f(f(\ell))}{f(\ell)}\max_{\lambda\in I}|\lambda_i| + \frac{1}{f(\ell)}\Big(\sum_{\substack{j\in J\\ j> f(f(\ell))}}|\lambda_i| + \sum_{i=1}^{\ell-\#J}\|E_i x\|_{n_i}\Big)\Big] \\
&\le \frac{1}{1-\frac{d}{\sqrt{f(m)}}}\Big[\frac{f(f(m))}{f(m)}\|z\|_s + \frac{1}{f(\ell)}\Big(\sum_{\substack{j\in J\\ j> f(f(\ell))}}|\lambda_i| + \sum_{i=1}^{\ell-\#J}\|E_i x\|_{n_i}\Big)\Big].
\end{aligned}$$

We can assume that $E_i\subset\operatorname{supp}(x)$ for $i\le\ell-\#J$. If $J\ne\emptyset$ then the cardinality of the support of $E_i x$ is smaller than the cardinality of $\operatorname{supp}(x)$. If $J=\emptyset$ and if, say, $E_1=\operatorname{supp}(x)$ then we could split E_1 into $\tilde E_1$ and $\tilde E_2$, choose $n_2=n_1$ and observe that above inequalities still hold. Thus we can assume that for all $i=1,\dots\ell-\#J$, the support of $E_i(x)$ is of lower cardinality than the support of x. Thus we can assume that our induction hypothesis applies to $E_i(x)$, $i\le\ell-\#J$.

We distinguish now between two cases.

If $\sum_{\substack{j\in J\\ j>f(f(\ell))}} |\lambda_j| \geq \frac{1}{\sqrt{G(\ell)}-1}\sum_{i=1}^{\ell-\#J}\|E_i(x)\|_{n_i}$, then it follows from Lemma 2.4 (applied to $\ell=m$ and $I=\{j\in J : j>f(f(\ell))\}$) that

$$\frac{1}{f(\ell)}\left(\sum_{\substack{j\in J\\ j>f(f(\ell))}}|\lambda_j| + \sum_{i=1}^{\ell-\#J}\|E_i x\|_{n_i}\right) \leq \frac{1}{f(\ell)}\sqrt{G(\ell)}\sum_{\substack{j\in J\\ j>f(f(\ell))}}|\lambda_j|$$

$$\leq \frac{1}{\sqrt{G(\ell)}}\left\|\sum_{\substack{j\in J\\ j>f(f(\ell))}}\lambda_j z_j\right\|_S \qquad (\#J\leq \ell)$$

$$\leq \frac{1}{\sqrt{G(m)}}\left\|\sum_{i\in I}\lambda_i z_i\right\|_S.$$

If $\sum_{\substack{j\in J\\ j>f(f(\ell))}} |\lambda_j| < \frac{1}{\sqrt{G(\ell)}-1}\sum_{i=1}^{\ell-\#J}\|E_i(x)\|_{n_i}$ we deduce that

$$\frac{1}{f(\ell)}\left(\sum_{\substack{j\in J\\ j\geq f(f(\ell))}}|\lambda_j| + \sum_{i=1}^{\ell-\#J}\|E_i(x)\|_{n_i}\right)$$

$$\leq \frac{1}{f(\ell)}\left(1+\frac{1}{\sqrt{G(\ell)}-1}\right)\sum_{i=1}^{\ell-\#J}\|E_i(x)\|_{n_i}$$

$$\leq \left(1+\frac{1}{\sqrt{G(m)}-1}\right)c(m^{f(m)})\frac{1}{\sqrt{G(m^{f(m)})}}\frac{1}{f(\ell)}\sum_{i=1}^{\ell-\#J}\|F_i(z)\|_S$$

(by the induction hypothesis applied to $E_i x$)

here we let $z=\sum_{i\in I}\lambda_i z_i$ and $F_i=\bigcup_{j\in E_i}\operatorname{supp}(z_j)$, and note that $F_i z=\sum_{j\in E_i}\lambda_j z_j$.

$$\leq \left(1+\frac{1}{\sqrt{G(m)}-1}\right)c(m^{f(m)})\frac{1}{\sqrt{G(m)}}\|z\|_S.$$

Thus in both cases we conclude that

$$\left\|\sum\lambda_i e_i\right\|_m \leq \left(\frac{1}{1-\frac{d}{f(m)}}\right)\left(\frac{f(f(m))\cdot\sqrt{G(m)}}{f(m)}+1+\frac{1}{\sqrt{G(m)}-1}\right)c(m^{f(m)})\frac{1}{\sqrt{G(m)}}\|z\|_s$$

$$= c(m)\frac{1}{\sqrt{G(m)}}\left\|\sum_{i\in I}\lambda_i z_i\right\|_S,$$

which finishes the proof of Lemma 2.6. □

We still have to prove Lemma 2.3 and Lemma 2.5.

Proof of Lemma 2.3. Note that for $r > 1$

$$G(r) \geq \widetilde{G}(r) := \sum_{j=1}^{\infty} 1_{[m_{j-1},m_j)}(r) \frac{f(r)f(k_j)}{f(rk_j)} \prod_{s=1}^{j-1} \frac{f(k_s)}{k_s} \qquad (m_0 = 1). \tag{18}$$

Therefore it will be enough to show that for $j \geq 2$,

$$\sum_{\ell\in\mathbb{N}, m_{j-1}\leq r_\ell < m_j} \frac{1}{\sqrt{\widetilde{G}(r_\ell)}} \leq \varepsilon_{j-1} + \varepsilon_j. \tag{19}$$

Using (9) we can easily prove by induction on $\ell \in \mathbb{N}_0$ that

$$r_\ell \geq r^{\left(\frac{3}{4}f(r)\right)^{2^\ell - 1}} \geq m_{j-1}^{\left(\frac{3}{4}f(m_{j-1})\right)^{2^\ell-1}} \tag{20}$$

and (by applying f) it follows from (10) that (recall that $m_{j-1} \geq m_1 \geq k_1$)

$$f(r_\ell) \geq \left(\frac{3}{4}f(m_{j-1})\right)^{2^\ell}. \tag{21}$$

We distinguish between two kinds of r_ℓ's in $\{r_\ell\colon\ m_{j-1} \leq r_\ell < m_j\}$. If $0 \leq \ell$ is such that $m_{j-1} \leq r_\ell \leq k_j$ we deduce that

$$\begin{aligned} \widetilde{G}(r_\ell) &= \frac{f(r_\ell)f(k_j)}{f(r_\ell k_j)} \prod_{s=1}^{j-1} \frac{f(k_s)}{k_s} \geq f(r_\ell)\frac{f(k_j)}{f(k_j^2)} \prod_{s=1}^{j-1} \frac{f(k_s)}{k_s} \\ &\geq \frac{1}{2} f(r_\ell) \prod_{s=1}^{j-1} \frac{f(k_s)}{k_s} \geq \frac{1}{2}\left(\frac{3}{4}f(m_{j-1})\right)^{2^\ell} \prod_{s=1}^{j-1} \frac{f(k_s)}{k_s} \qquad \text{(by (21))} \\ &\geq \frac{3}{16}k_{j-1}\left(\frac{3}{8}k_{j-1}\right)^{2^\ell-1} \prod_{s=1}^{j-1}\frac{f(k_s)}{k_s} \end{aligned} \tag{22}$$

(note that $f(m_{j-1}) \geq f(m_{j-1}k_{j-1}) - f(k_{j-1}) \geq \frac{1}{2}k_{j-1}$)

$$= \frac{3}{16} f(k_{j-1}) \left(\frac{3}{8}k_{j-1}\right)^{2^\ell-1} \prod_{s=1}^{j-2} \frac{f(k_s)}{k_s} \geq \frac{3}{16} f(k_{j-1}) \left(\frac{3}{8}k_1\right)^{\ell} \prod_{s=1}^{j-2} \frac{f(k_s)}{k_s}.$$

Thus

$$\sum_{\substack{\ell\in\mathbb{N}_j \\ m_{j-1}\leq r_\ell\leq k_j}} \frac{1}{\sqrt{\widetilde{G}(r_\ell)}} \leq \left(\frac{16}{3}\frac{1}{f(k_{j-1})} \prod_{s=1}^{j-2}\frac{k_s}{f(k_s)}\right)^{1/2} \sum_{\ell=0}^{\infty}\left(\frac{8}{3}k_1\right)^{-\ell/2} < \varepsilon_{j-1} \qquad \text{(by (11))}. \tag{23}$$

To estimate $1/\sqrt{\widetilde{G}(r_\ell)}$, with $k_j < r_\ell < m_j$, we put $\ell_0 = \min\{\ell \mid r_{\ell_0} > k_j\}$ and $\ell_1 = \max\{\ell \mid r_\ell < m_j\}$ and observe for $\ell \in \{\ell_0, \ell_0+1, \dots, \ell_1\}$ that

$$\widetilde{G}(r_\ell) = \frac{f(r_\ell)f(k_j)}{f(r_\ell k_j)} \prod_{s=1}^{j-1}\frac{f(k_s)}{k_s} \geq f(k_j)\frac{f(r_\ell)}{f(r_\ell^2)} \prod_{s=1}^{j-1}\frac{f(k_s)}{k_s} \geq \frac{1}{2}f(k_j)\prod_{s=1}^{j-1}\frac{f(k_s)}{k_s}. \tag{24}$$

As in (20) and (21) we observe that

$$r_{\ell_0}^{\left(\frac{3}{4}f(r_{\ell_0})\right)^{2^{\ell_1-\ell_0-1}}} \leq r_{\ell_1} < m_j \text{ and } \left(\frac{3}{4}f(r_{\ell_0})\right)^{2^{\ell_1-\ell_0}} \leq f(m_j) \leq k_j$$

which implies by (10) that $2^{\ell_1-\ell_0} f(\frac{3}{4}f(r_{\ell_0})) \leq \frac{4}{3}f(k_j)$ and thus (by (9))

$$\ell_1 - \ell_0 \leq \log_2 \frac{f(k_j)}{\frac{3}{4}f\left(\frac{3}{4}f(r_{\ell_0})\right)} \leq \log_2 f(k_j). \tag{25}$$

The inequalities (24), (25) and (11) imply that

$$\begin{aligned}\sum_{\ell\in\mathbb{N}, k_j<r_\ell<m_j} \widetilde{G}(r_\ell)^{-1/2} &\leq 2(\ell_1-\ell_0)\left(\frac{1}{f(k_j)}\prod_{s=1}^{j-1}\frac{k_s}{f(k_s)}\right)^{1/2} \\ &\leq 2\frac{\log_2 f(k_j)}{\sqrt{f(k_j)}}\left(\prod_{s=1}^{j-1}\frac{k_s}{f(k_s)}\right)^{1/2} \leq \varepsilon_j.\end{aligned} \tag{26}$$

Finally (23) and (26) imply the claimed inequality (19). □

Proof of Lemma 2.5. Since f is close to a logarithmic function we can choose a $c > 1$ so that for $\xi, \xi', R \geq 1$ and $0 < r < 1$

$$\frac{1}{c}(f(\xi)+f(\xi')) \leq f(\xi\cdot\xi') \leq f(\xi)+f(\xi') \tag{27}$$

$$\frac{1}{c}Rf(\xi) \leq f(\xi^R) \leq Rf(\xi) \tag{28}$$

$$rf(\xi) \leq f(\xi^r) \tag{29}$$

$$\frac{1}{c}f(\xi) \leq f(\xi) - 1 \quad \text{if} \quad \xi \geq 2. \tag{30}$$

Assuming furthermore that $c \geq 2$ we deduce for $\xi \geq 1$ that

$$f(\xi^{1/\sqrt{f(\xi)}}) = \log_2(\xi^{1/\sqrt{f(\xi)}}+1) \leq 1 + \frac{1}{\sqrt{f(\xi)}}\log_2(\xi) \leq 1+\sqrt{f(\xi)} \leq c\sqrt{f(\xi)},.$$

Let $d = 4c^3$, let $r \in \mathbb{R}$, such that $f(r) > d^2$, and $x \in c_{00}$. Choose $r \leq \ell < \infty$ and $E_1 < E_2 < \cdots < E_\ell$ so that $|||x|||_r = \|x\|_\ell = \frac{1}{f(\ell)}\sum_{i=1}^{\ell}\|E_i(x)\|_S$. For $i \in \{1,2,\ldots,\ell\}$ let $n_i \in \mathbb{N}\cup\{\infty\}$ so that $||E_ix\|_S = \|E_ix\|_{n_i}$. For two numbers $\tilde{r}, \widetilde{R}$, $2 \leq \tilde{r} < \widetilde{R} < \infty$, we let $M := M(\tilde{r}, \widetilde{R}) := \{i \in \ell\colon\ \tilde{r} \leq n_i < \widetilde{R}\}$ and we choose for $i \in M$, $E_1^{(i)} < \cdots < E_{n_i}^{(i)}$, with $E_j^{(i)} \subset E_i$ whenever $j \leq n_i$ so that $\|E_ix\|_S = \frac{1}{f(n_i)}\sum_{j=1}^{n_i}\|E_j^{(i)}x\|_S$. Now we observe that $\{E_i\colon\ i \notin M\} \cup \bigcup_{i\in M}\{E_j^{(i)}\colon 1 \leq j \leq n_i\}$ is well ordered by $<$ and its cardinality is

$\ell - \#M + \sum_{i\in M} n_i$ which is at least ℓ and at most $\widetilde{R}\ell$. Thus we deduce

$$\begin{aligned}
|||x|||_r = \frac{1}{f(\ell)}\sum_{i=1}^{\ell}\|E_i x\|_S &\geq \frac{1}{f\left(\ell - \#M + \sum\limits_{i\in M} n_i\right)}\left[\sum_{\substack{i=1\\ i\notin M}}^{\ell}\|E_i x\|_s + \sum_{i\in M}\sum_{j=1}^{n_i}\|E_j^{(i)}x\|_S\right]\\
&\geq \frac{1}{f(\ell\cdot\widetilde{R})}\left[\sum_{i=1, i\notin M}^{\ell}\|E_i x\|_S + \sum_{i\in M} f(n_i)\|E_i x\|_S\right] \text{ (since } n_i \leq \widetilde{R} \text{ for } i\in M)\\
&\geq \frac{1}{f(\ell\cdot\widetilde{R})}\left[\sum_{i=1}^{\ell}\|E_i x\|_S + \sum_{i\in M}(f(\tilde{r})-1)\|E_i x\|_S\right]\\
&\geq \frac{1}{f(\ell\cdot\widetilde{R})}\left[\sum_{i=1}^{\ell}\|E_i x\|_S + \frac{f(\tilde{r})}{c}\sum_{i\in M}\|E_i x\|_S\right].
\end{aligned}$$

Solving these inequalities for $\frac{1}{f(\ell)}\sum\limits_{i\in M}\|E_i x\|_S$ we obtain that

$$\begin{aligned}
\frac{1}{f(\ell)}\sum_{i\in M}\|E_i x\|_S &\leq \frac{1}{f(\ell)}\left[\frac{1}{f(\ell)} - \frac{1}{f(\ell\cdot\widetilde{R})}\right]\frac{cf(\ell\widetilde{R})}{f(\tilde{r})}\sum_{i=1}^{\ell}||E_i x\|_S\\
&= \frac{c}{f(\tilde{r})}\left[\frac{f(\ell\cdot\widetilde{R}) - f(\ell)}{f(\ell)}\right]|||x|||_r\\
&\leq c\frac{f(\widetilde{R})}{f(\tilde{r})f(\ell)}|||x|||_r \leq c\frac{f(\widetilde{R})}{f(\tilde{r})f(r)}|||x|||_r.
\end{aligned}$$

Choosing for the $(\tilde{r}, \widetilde{R})$ the values $(2, r^{1/\sqrt{f(r)}})$, $(r^{1/\sqrt{f(r)}}, r)$, $(r, r^{\sqrt{f(r)}})$ and $(r^{\sqrt{f(r)}}, r^{f(r)})$ we deduce for all choices $\frac{f(\widetilde{R})}{f(\tilde{r})f(r)} \leq \frac{c^2}{\sqrt{f(r)}}$. This implies that

$$\frac{1}{f(\ell)}\sum_{\substack{1\leq i\leq \ell\\ n_i < r^{f(r)}}}\|E_i x\| \leq \frac{4c^3}{\sqrt{f(r)}}|||x|||_r$$

and thus that

$$|||x|||_r = \frac{1}{f(\ell)}\left[\sum_{\substack{i=1\\ n_i<r^{f(r)}}}^{\ell}\|E_i x\|_{n_i} + \sum_{\substack{i=1\\ n_i>r^{f(r)}}}^{\ell}\|E_i x\|_{n_i}\right] \leq \frac{d}{\sqrt{f(r)}}|||x|||_r + \frac{1}{f(\ell)}\sum_{\substack{i=1\\ n_i>r^{f(r)}}}^{\ell}\|E_i x\|.$$

Now let $I = \{i \leq \ell \mid n_i = \infty \text{ or } n_i \leq r^{f(r)}\}$ and choose for $i \in I$ a $j_i \in \operatorname{supp}(x) \cap E_i$ so that $\|E_i x\|_{\ell_\infty} = |x_{j_i}|$, and reorder $(n_i)_{i\in\{1\ldots\ell\}\setminus I}$ and $(E_i)_{i\in\{1\ldots\ell\}\setminus I}$ into $\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_{\ell-\#I}$ and $\widetilde{E}_1 < \widetilde{E}_2 < \cdots < \widetilde{E}_{\ell-\#I}$. Letting $J = \{j_i\colon\ i \in I\}$ we obtain the claimed inequality:

$$|||x|||_r \leq \frac{1}{1 - \frac{d}{\sqrt{f(r)}}}\frac{1}{f(\ell)}\left[\sum_{j\in J}|x_j| + \sum_{i=1}^{\ell-\#J}\|\widetilde{E}_i x\|_{\tilde{n}_i}\right].$$

□

3. The construction of strictly singular non-compact operators on GM

The goal of this section is to prove Theorem 1.1. We postpone the definition of the space GM and only state some properties needed for the proof of Theorem 1.1.

Recall that the *spreading model* of a seminormalized weakly null sequence (x_i) in some Banach space $(X, \|\cdot\|)$ is a sequence (y_i) along with a norm $|\cdot|$ on the span of (y_i) such that

$$\left|\sum_{i=1}^k \lambda_i y_i\right| = \lim_{\substack{N\to\infty \\ N\le n_1<n_2<\cdots n_k}} \|\sum_{i=1}^k \lambda_i x_i\|$$

for every finite sequence of scalars $(\lambda_i)_{i=1}^k$.

Proposition 3.1. *The spreading model of the unit vector basis of GM is the unit vector basis of S.*

In order to defining GM a lacunary set $J \subseteq \mathbb{N}$ is used which has the property that if $n, m \in J$ and $n < m$ then $4n^2 \le \log\log\log m$, and $f(\min J) \ge 256$.

Proposition 3.2. *The norm of GM satisfies*

$$\|x\|_S \le \|x\|_{GM} \le \|x\|_S + \sum_{\ell\in J} \|x\|_\ell, \text{ for } x \in c_{00} \tag{32}$$

(here $\|\cdot\|_\ell$ still denotes the equivalent norm on S introduced at the end of Section 1).

Propositions 3.1 and 3.2 will be shown at end of this section. The first inequality in (32) is an immediate consequence of the definition of GM and the second inequality will follow from Lemma 3.3.

Recall from the previous section that there exists a seminormalized block sequence (x_i^*) in S^* and a non-decreasing function $C : [2,\infty) \to (0,\infty)$ which satisfies (1). Secondly we observe that from the remark following Lemma 2.6 and the conditions on J it follows that

$$\sum_{\ell\in J} \frac{1}{C(\ell)} < \infty. \tag{33}$$

The sequence (x_i^*) may not be seminormalized in GM^*, i.e. (x_i^*) could be a null sequence in GM (which would imply that the operator T defined in Section 2 would be compact on GM). But Proposition 3.1 implies that we can replace each x_i^* by $\tilde{x}_i^*$ which has the same distribution and up to some arbitrarily small number $\varepsilon > 0$ the same norm in GM^* as in S^*. Indeed, if $z^* = \sum_{i=1}^k \lambda_i e_i^* \in S^* \cap c_{00}$ and $z = \sum_{i=1}^k \mu_i e_i \in S$, with $\|z\|_S = 1$ and $z^*(z) = \|z\|_{S^*}$ it follows that

$$\begin{aligned}\|z^*\|_S = \sum_{i=1}^k \lambda_i\mu_i &\le \liminf_{\substack{N\to\infty \\ N\le n_1<n_2<\cdots n_k}} \left\|\sum_{i=1}^k \lambda_i e_{n_i}\right\|_{GM} \cdot \left\|\sum_{i=1}^k \mu_i e^*_{n_i}\right\|_{GM^*} \\ &= \liminf_{\substack{N\to\infty \\ N\le n_1<n_2<\cdots n_k}} \left\|\sum_{i=1}^k \mu_i e^*_{n_i}\right\|_{GM^*}.\end{aligned}$$

Thus we will assume from now on that (x_i^*) is also seminormalized in GM^*.

Then define the operator $T : GM \to GM$ by $T = \sum_i x_i^* \otimes e_i$. We now present the

Proof of Theorem 1.1 We first show that T is bounded. Every $x \in GM$ can be written as $x = \sum \lambda_i z_i$ with $x^*_{i-1} < z_i < x^*_{i+1}$, and $x_i^*(z_i) = 1$ for all i. Thus $Tx = \sum \lambda_i e_i$. By (32) we have that for any $x \in c_{00}$,

$$\begin{aligned} \|Tx\|_{GM} &\leq \|Tx\|_S + \sum_{\ell\in J} \|\sum \lambda_i e_i\|_\ell \\ &\leq \|Tx\|_S + \sum_{\ell\in J} \frac{1}{C(\ell)} \|\sum \lambda_i z_i\|_S \text{ (by property (1))} \\ &\leq \left(\|T : S \to S\| + \sum_{\ell\in J} \frac{1}{C(\ell)} \right) \|x\|_S. \\ &\leq \left(\|T : S \to S\| + \sum_{\ell\in J} \frac{1}{C(\ell)} \right) \|x\|_{GM}. \end{aligned} \tag{34}$$

Now we show that T is strictly singular. From the definition of (x_i^*) it follows that T has an infinite dimensional kernel. Since T can be written as $T = \lambda + \tilde{T}$ with $\tilde{T}$ being strictly singular it follows that $\lambda = 0$.

Finally, T is not a compact operator since (x_i^*) is seminormalized in GM^*. □

Let X be either the space S or GM. For any sequence $\nu = (\nu_i) \in \ell_\infty$ define the operator

$$T_\nu = \sum_i \nu_i x_i^* \otimes e_i : X \to GM.$$

The above proof shows that T_ν is a bounded strictly singular operator with

$$\|\nu\|_\infty \inf_i \|x_i^*\|_X \leq \|T_\nu : X \to GM\| \leq \left(\|T : S \to S\| + \sum_{\ell\in J} \frac{1}{C(\ell)} \right) \|\nu\|_\infty.$$

Therefore ℓ_∞ embeds in the space of operators from X to GM. If $\nu \in \ell_\infty \backslash c_0$ then $T_\nu : X \to GM$ is non-compact.

Let us now recall the definition of the space GM.

Let $\mathbf{Q}$ be the set of scalar sequences with finite support and rational coordinates whose absolute value is at most one. Write J (introduced above) in increasing order as $\{j_1, j_2, \ldots\}$. Now let $K \subset J$ be the set $\{j_1, j_3, j_5, \cdots\}$ and $L \subset J$ be the set $\{j_2, j_4, j_6, \ldots\}$. Let σ be an injective function from the collection of all finite sequences of elements of $\mathbf{Q}$ to L such that if $z_1, \ldots, z_i$ is such a sequence, then $(1/20) f(\sigma(z_1, \ldots, z_i)^{1/40}) \geq \#\mathrm{supp}\,(\sum_{j=1}^i z_j)$. Then, recursively, we define a set of functionals of the unit ball of GM^* as follows: Let

$$GM_0^* = \{\lambda e_n^* : n \in \mathbb{N},\ |\lambda| \leq 1\}.$$

Assume that GM_k^* has been defined. Then GM_{k+1}^* is the set of all functionals of the form $E\ z^*$ where $E \subseteq \mathbb{N}$ is an interval and z^* has one of the following three forms:

$$z^* = \sum_{i=1}^{\ell} \alpha_i z_i^* \tag{35}$$

where $\sum_{i=1}^{\ell} |\alpha_i| \leq 1$ and $z_i^* \in GM_k^*$ for $i = 1, \ldots, \ell$.

$$z^* = \frac{1}{f(\ell)} \sum_{i=1}^{\ell} z_i^* \tag{36}$$

where $z_i^* \in GM_k^*$ for $i = 1, \dots, \ell$, and $z_1^* < \cdots < z_\ell^*$.

$$z^* = \frac{1}{\sqrt{f(\ell)}} \sum_{i=1}^{\ell} z_i^* \text{ and } z_i^* = \frac{1}{f(m_i)} \sum_{j=1}^{m_i} z_{i,j}^* \tag{37}$$

where $z_{i,j}^* \in GM_k^*$ for $1 \le i \le \ell$ and $1 \le j \le m_i$, $z_{1,1}^* < \cdots < z_{1,m_1}^* < z_{2,1}^* < \cdots < z_{\ell,m_\ell}^*$, $m_1 = j_{2\ell}$, z_i^* has rational coordinates, and $m_{i+1} = \sigma(z_1^*, \dots, z_i^*)$, for $i = 1, \dots, \ell - 1$.

Finally, the norm of GM is defined by

$$\|x\|_{GM} = \sup\{z^*(x) : z^* \in \cup_{k=0}^{\infty} GM_k^*\}.$$

For an interval $I \subseteq \mathbb{N}$ we define

$$J(I) = \{\sigma(x_1^*, x_2^*, \dots, x_n^*) : n \in \mathbb{N},\ x_1^* < x_2^* < \cdots < x_n^*,\ \min I \le \max \operatorname{supp}(x_n^*) < \max I\}.$$

Also, for $x^* \in GM^*$ we define $J(x^*) = J([\min \operatorname{supp}(x^*), \max \operatorname{supp}(x^*)])$.

The next result relates the functionals of the unit ball of GM^* to the functionals of the unit ball of S^*.

Lemma 3.3. *For any $z^* \in \cup_{k=0}^{\infty} GM_k$ there exist $T_0(z^*) \in \mathrm{Ba}\,(\mathrm{S}^*)$, the unit ball of S^*, and a family $(T_j(z^*))_{j \in J(z^*)} \subset \mathrm{Ba}\, S^*$ such that*

1) *For $j \in \{0\} \cup J(z^*)$, $\operatorname{supp} T_j(z^*) \subseteq [\min \operatorname{supp} z^*, \max \operatorname{supp} z^*]$.*
2) *For $j \in J(z^*)$*

$$T_j(z^*) \in \mathrm{aco}\,\Big\{\frac{1}{f(j)} : \sum_{s=1}^{j} x_s^* : x_1^* < \cdots < x_j^* \text{ are in } \mathrm{Ba}\,(S^*)\Big\}.$$

where "aco" denotes the absolute convex hull.

3)

$$z^* = T_0(z^*) + \sum_{j \in J(z^*)} T_j(z^*).$$

Proof. We proceed by induction on k (assume that $z^* \in GM_k^*$). For $k = 0$, $z^* = \lambda e_n^*$ for some $\lambda \in \mathbb{R}$, $|\lambda| \le 1$ and some $n \in \mathbb{N}$. Then $J(z^*) = \emptyset$ and $T_0(z^*) = z^*$. The inductive step, from k to $k+1$ proceeds as follows: By the definition of GM_{k+1}^*, we separate three cases:

Case 1: Assume that $z^* = E(\sum_{i=1}^{\ell} \alpha_i z_i^*)$ where $E \subset \mathbb{N}$ is an interval, $z_i^* \in GM_k^*$ for all $i \le \ell$ and $\sum_{i=1}^{\ell} |\alpha_i| \le 1$. Let $\tilde{E} = [\min \operatorname{supp} E(z^*), \max \operatorname{supp} E(z^*)]$. Then $\tilde{E} \subseteq E$ and by the induction hypothesis we have

$$z^* = \tilde{E}\left(\sum_{i=1}^{\ell} \alpha_i z_i^*\right) = \sum_{i=1}^{\ell} \alpha_i \tilde{E}(z_i^*) = \sum_{i=1}^{\ell} \alpha_i T_0(\tilde{E}(z_i^*)) + \sum_{i=1}^{\ell} \sum_{j \in J(\tilde{E}(z_i^*))} T_j(\tilde{E}(z_i^*)).$$

Set

$$T_0(z^*) = \sum_{i=1}^{\ell} \alpha_i T_0(\tilde{E}(z_i^*))$$

and after noting that $\cup_{i=1}^{\ell} J(\tilde{E}(z_i^*)) \subseteq J(z^*)$, for $j \in J(z^*)$ set

$$T_j(z^*) = \sum_{\substack{i=1,\dots,\ell \\ j \in J(\tilde{E}(z_i^*)}} \alpha_i T_j(\tilde{E}(z_i^*))$$

where the sum over an empty set of indices is zero, and $T_j(z^*) = 0$ if $j \in J(z^*)\backslash\cup_{i=1}^{\ell} J(\tilde{E}(z_i^*))$. It is easy to see that the above choices of $T_0(z^*)$ and $T_j(z^*)$, $j \in J(z^*)$, satisfy the conclusion of the lemma.

Case 2: Assume that $z^* = E\left(\frac{1}{f(\ell)}\sum_{i=1}^{\ell} z_i^*\right)$ where $E \subseteq \mathbb{N}$ is an interval and $z_1^* < z_2^* < \cdots < z_\ell^*$ in GM_k^*. By the induction hypothesis we have that

$$z^* = \frac{1}{f(\ell)}\sum_{i=1}^{\ell} E(z_i^*) = \frac{1}{f(\ell)}\sum_{i=1}^{\ell} T_0(Ez_i^*) + \sum_{i=1}^{\ell}\sum_{j\in J(E(z_i^*))}\frac{1}{f(\ell)}T_j(Ez_i^*).$$

Set

$$T_0(z^*) = \frac{1}{f(\ell)}\sum_{i=1}^{\ell} T_0(Ez_i^*)$$

and after noting that $J(E(z_i^*)) \cap J(E(z_j^*)) = \emptyset$ (since σ is injective) for $1 \le i \ne j \le \ell$, and $\cup_{i=1}^{\ell} J(E(z_i^*)) \subseteq J(z^*)$, for $j \in J(z^*)$ set

$$T_j(z^*) = \frac{1}{f(\ell)}T_j(Ez_i^*) \text{ if } j \in J(E(z_i^*)),$$

and $T_j(z^*) = 0$ if $j \in J(z^*)\backslash\cup_{i=1}^{\ell} J(E(z_i^*))$. It is easy to see that the conclusion of the lemma is satisfied.

Case 3: Assume that $z^* = E\left(\frac{1}{\sqrt{f(\ell)}}\sum_{i=1}^{\ell} z_i^*\right)$ and $z_i^* = \frac{1}{f(m_i)}\sum_{j=1}^{m_i} z_{i,j}^*$ where $z_{i,j}^* \in GM_k^*$ for $1 \le i \le \ell$ and $1 \le j \le m_i$, $z_{1,1}^* < \cdots < z_{1,m_1}^* < z_{2,1}^* < \cdots < z_{\ell,m_\ell}^*$, $m_1 = j_{2\ell}$, z_i^* has rational coordinates, and $m_{i+1} = \sigma(z_1^*, \ldots, z_i^*)$, for $i = 1, \ldots, \ell - 1$. Let

$$i_1 = \min\big\{i \in \{1, \ldots, \ell\} : E \cap \operatorname{supp}(z_i^*) \ne \emptyset\big\}$$

and

$$i_2 = \max\big\{i \in \{1, \ldots, \ell\} : E \cap \operatorname{supp}(z_i^*) \ne \emptyset\big\}.$$

If $i_1 = i_2$ then we proceed as in Case 2. Therefore without loss of generality, we assume that $i_1 < i_2$. Let

$$j_1 = \min\big\{j \in \{1, \ldots, m_{i_1}\} : E \cap \operatorname{supp}(z_{i_1,j}^*) \ne \emptyset\big\}$$

and

$$j_2 = \max\big\{j \in \{1, \ldots, m_{i_2}\} : E \cap \operatorname{supp}(z_{i_2,j}^*) \ne \emptyset\big\}$$

By the induction hypothesis we have

$$\begin{aligned}
E(z^*) &= \frac{1}{\sqrt{f(\ell)}}\left[\frac{1}{f(m_{i_1})}\sum_{j=j_1}^{m_{i_1}} E(z^*_{i_1,j}) + \sum_{i=i_1+1}^{i_2-1}\frac{1}{f(m_i)}\sum_{j=1}^{m_i} z^*_{i,j} + \frac{1}{f(m_{i_2})}\sum_{j=1}^{j_2} E(z^*_{i_2,j})\right] \\
&= \frac{1}{\sqrt{f(\ell)}}\frac{1}{f(m_{i_1})}\sum_{j=j_1}^{m_{i_1}} T_0(E(z^*_{i_1,j})) \\
&\quad + \sum_{i=i_1+1}^{i_2-1}\frac{1}{\sqrt{f(\ell)}}\frac{1}{f(m_i)}\sum_{j=1}^{m_i} T_0(Ez^*_{i,j}) + \frac{1}{\sqrt{f(\ell)}}\frac{1}{f(m_{i_2})}\sum_{j=1}^{j_2} T_0(Ez^*_{i_2,j}) \\
&\quad + \sum_{j=j_1}^{m_{i_1}}\sum_{k\in J(Ez^*_{i_1,j})}\frac{1}{\sqrt{f(\ell)}}\frac{1}{f(m_{i_1})}T_k(Ez^*_{i_1,j}) \\
&\quad + \sum_{i=i_1+1}^{i_2-1}\sum_{j=1}^{m_i}\sum_{k\in J(Ez^*_{i,j})}\frac{1}{\sqrt{f(\ell)}}\frac{1}{f(m_i)}T_k(Ez^*_{i,j}) \\
&\quad + \sum_{j=1}^{j_2}\sum_{k\in J(Ez^*_{i_2,j})}\frac{1}{\sqrt{f(\ell)}}\frac{1}{f(m_{i_2})}T_k(Ez^*_{i_2,j})
\end{aligned}$$

Set

$$T_0(Ez^*) = \frac{1}{\sqrt{f(\ell)}}\frac{1}{f(m_{i_1})}\sum_{j=j_1}^{m_{i_1}} T_0(E(z^*_{i_1,j}))$$

and after noting that

$$\{m_{i_1+1},\ldots,m_{i_2}\}\cup\cup_{j=j_1}^{m_{i_1}} J(Ez^*_{i_1,j})\cup\cup_{i=i_1+1}^{i_2-1}\cup_{j=1}^{m_i} J(Ez^*_{i,j})\cup\cup_{j=1}^{j_2} J(Ez^*_{i_2,j})\subseteq J(Ez^*) \tag{38}$$

and that the sets $\{m_{i_1+1},\ldots,m_{i_2}\}$, $J(Ez^*_{i_1,j})$ (for $j=j_1,\ldots,m_{i_1}$), $J(Ez^*_{i,j})$ (for $i=i_1+1,\ldots,i_2-1$ and $j=1,\ldots,m_i$), $J(Ez^*_{i_2,j})$ (for $j=1,\ldots,j_2$) are mutually disjoint (by the injectivity of σ), set

$$T_k(Ez^*) = \begin{cases}
\frac{1}{\sqrt{f(\ell)}}\frac{1}{f(m_i)}\sum_{j=1}^{k} T_0(Ez^*_{i,j}) & \text{if } k=m_i\in\{m_{i_1+1},\ldots,m_{i_2-1}\} \\
\frac{1}{\sqrt{f(\ell)}}\frac{1}{f(m_{i_2})}\sum_{j=1}^{j_2} T_0(Ez^*_{i_2,j}) & \text{if } k=m_{i_2} \\
\frac{1}{\sqrt{f(\ell)}}\frac{1}{f(m_{i_1})}T_k(Ez^*_{i_1,j}) & \text{if } k\in\cup_{i=j_1}^{m_{i_1}} J(Ez^*_{i_1,j}) \\
\frac{1}{\sqrt{f(\ell)}}\frac{1}{f(m_i)}T_k(Ez^*_{i,j}) & \text{if } k\in\cup_{i=i_1+1}^{i_2-1}\cup_{j=1}^{m_i} J(z^*_{i,j}) \\
\frac{1}{\sqrt{f(\ell)}}\frac{1}{f(m_{i_2})}T_k(Ez^*_{i_2,j}) & \text{if } k\in\cup_{j=1}^{j_2} J(Ez^*_{i_2,j}) \\
0 & \text{if } k\in J(Ez^*) \text{ otherwise.}
\end{cases}$$

It is easy to see that the conclusion of the lemma is satisfied. ☐

Note that inequality (32) (which was used in the proof of Theorem 1.1) is an immediate consequence of Lemma 3.3. It only remains to give the

Proof of Proposition 3.1. Let $x = \sum_{i=1}^k \lambda_i e_i \in c_{00}$. We want to show that

$$\lim_{\substack{N\to\infty \\ N\le n_1<n_2<\cdots<n_k}} \|\sum_{i=1}^k \lambda_i e_{n_i}\|_{GM} = \|\sum_{i=1}^k \lambda_i e_i\|_S. \tag{39}$$

Let $\varepsilon > 0$. Since J is lacunary enough and σ is injective we can choose $N \in \mathbb{N}$ sufficiently large, such that

$$k \max_i |\lambda_i| \sum_{\ell\in J([N,\infty))} \frac{1}{f(\ell)} < \varepsilon. \tag{40}$$

Thus, if $N \le n_1 < n_2 < \cdots n_k$, then by Lemma 3.3,

$$\begin{aligned}
\|\sum_{i=1}^k \lambda_i e_{n_i}\|_{GM} &\le \|\sum_{i=1}^k \lambda_i e_i\|_S + \sum_{\ell\in J([N,\infty))} \|\sum_{i=1}^k \lambda_i e_{n_i}\|_\ell \\
&\le \|\sum_{i=1}^k \lambda_i e_i\|_S + \sum_{i=1}^k |\lambda_i| \sum_{\ell\in J([N,\infty))} \|e_{n_i}\|_\ell \\
&\le \|\sum_{i=1}^k \lambda_i e_i\|_S + k \max_i |\lambda_i| \sum_{\ell\in J([N,\infty))} \frac{1}{f(\ell)} \le \|\sum_{i=1}^k \lambda_i e_i\|_S + \varepsilon \text{ (by (40))}
\end{aligned}$$

which finishes the proof of (39). □

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