Σ smooth, oriented closed surface.

last time:

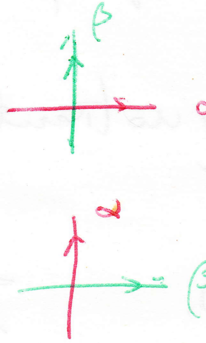
$\bar{\mathcal{L}}(\Sigma) = \mathbb{Z}\{$free homotopy class $\bar{\alpha}$ of unoriented loops on $\Sigma\}$

从p开始先走α, 再走β^{-1}.

$$[\bar{\alpha}, \bar{\beta}] = \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha.\beta)\left(\overline{\alpha_p \beta} - \overline{\alpha_p \beta^{-1}}\right)$$

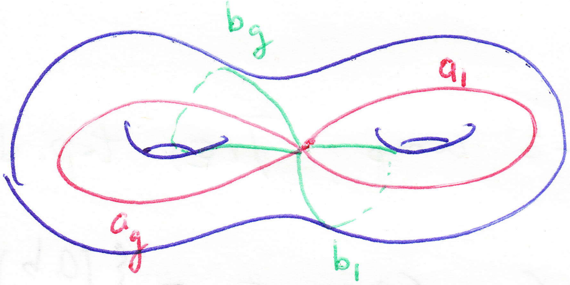
← 从p开始先走α, 再走β.

$$\varepsilon(p; \alpha.\beta) = \begin{cases} 1 \\ -1 \end{cases}$$



Thm $(\bar{\mathcal{L}}(\Sigma), [\,,\,])$ is a well-defined Lie algebra.

Character variety



- $\pi = \pi_1(\Sigma) = \langle a_1, b_1, \cdots, a_g, b_g \mid \underbrace{a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1}_{R} \rangle$

G Lie group. $\underline{\mathrm{Hom}(\pi, G)} = \{\rho : \pi \to G \text{ gr. hom.}\}$ w/ compact-open topology

representation variety.

Rm

- If G is algebraic, then $Hom(\pi, G)$ is an affine algebraic variety, since $Hom(\pi, G) \subset G^{2g}$ defined by one polynomial relation given by R.

- G acts on $Hom(\pi, G)$ by conjugation

$$(g \circ \rho)(x) = g \cdot \rho(x) \cdot g^{-1}, \ \forall x \in \pi, \ g \in G.$$

Denote the quotient by $Hom(\pi, G)/G$.

- $G = PSL(2, \mathbb{R})$, $\quad \mathcal{T}(\Sigma) \subset Hom(\pi, G)/G$

$\uparrow$
Teichmüller space.

$$\mathcal{T}(\Sigma) = \{\rho : \pi \to G \mid \rho \text{ discrete, faithful}\}/\text{conj}$$

- $Hom(\pi, G)/G$ $4g-3$ comp, $\mathcal{T}(\Sigma)$ top. dim. comp

- In this lecture, $G = SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} ad-bc=1 \\ a,b,c,d \\ \in \mathbb{C} \end{array} \right\}$.

Goal: define a Poisson structure on $Hom(\pi, G)/G$ with an $(\mathcal{L}, [\,,\,])$ Poisson action.

<u>Def</u>: X smooth mfd (w/ singularities), $C^\infty(X)$ ring of smooth functions on X. A bilinear map $\{,\}: C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ is a <u>Poisson bracket</u> if $\forall f, g, h \in C^\infty(X)$,

1. $\{f, g\} = -\{g, f\}$,
2. $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$,
3. $\{fg, h\} = f\{g, h\} + g\{f, h\}$.

- 1 + 2 : $\{.\}$ is a Lie bracket.
- 3 : $\{.\}$ distributive over $\cdot$.

Rm: also consider subring of $C^\infty(X)$, e.g. ring of regular functions $R(X)$ of X, if X is algebraic.

<u>Def</u>: $(L, [,])$ Lie algebra, $(X, \{,\})$ Poisson.

A <u>Poisson action</u> of $(L, [,])$ on $(X, \{,\})$ is a Lie algebra homomorphism

$$\phi: (L, [,]) \longrightarrow (C^\infty(X), \{,\}).$$

To define the Poisson structure, it's to consider the functions on $\mathrm{Hom}(\pi, G)/G$.

Trace functions:

$\alpha \in \pi$, $\quad tr_\alpha : \mathrm{Hom}(\pi, G)/G \longrightarrow \mathbb{C}$

$$[\rho] \longmapsto tr\, \rho(\alpha)$$

It is well defined, since

$$\begin{aligned} tr_\alpha [g \circ \rho] &= tr\, (g \circ \rho)(\alpha) \\ &= tr\; g \cdot \rho(\alpha) \cdot g^{-1} \\ &= tr\, \rho(\alpha) \\ &= tr_\alpha [\rho], \quad \forall g \in G. \end{aligned}$$

It depends only on the conjugacy class of α, since

$$\begin{aligned} tr_{\beta\alpha\beta^{-1}}(\rho) &= tr\, \rho(\beta \cdot \alpha \cdot \beta^{-1}) \\ &= tr\, \rho(\beta) \cdot \rho(\alpha) \cdot \rho(\beta)^{-1} \\ &= tr\, \rho(\alpha) \\ &= tr_\alpha(\rho), \quad \forall \rho \in \mathrm{Hom}(\pi, G), \\ &\qquad \alpha, \beta \in \pi. \end{aligned}$$

$\mathcal{R}(\Sigma) = \mathrm{Span}_{\mathbb{C}} \{ tr_\alpha \mid \alpha \in \pi \}$

Prop: $\mathcal{R}(\Sigma)$ is a subring of $C^\infty(\mathrm{Hom}(\pi, G)/G)$

Pf: $\forall A, B \in SL(2, \mathbb{C})$, we have

$$tr A \cdot tr B = tr AB + tr AB^{-1}.$$

As a consequence, $\forall \alpha, \beta \in \pi$,

$$\underline{tr_\alpha \cdot tr_\beta = tr_{\alpha\beta} + tr_{\alpha\beta^{-1}}}. \quad \cancel{\forall \alpha, \beta \in \pi} \quad \square$$

trace identity!

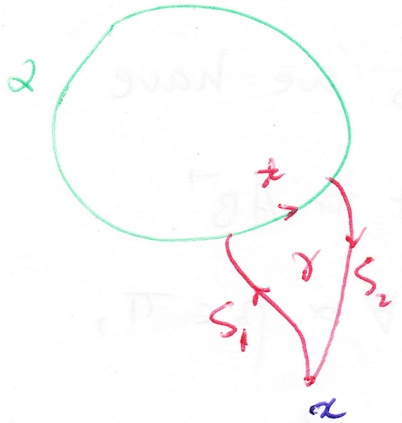
<u>Algebraic nonsense</u>

Thm. $\mathcal{R}(\Sigma)$ is the ring of G-invariant regular functions on $\mathrm{Hom}(\pi, G)$, hence is the coordinate ring (ring of regular functions) of the <u>character variety</u>

$$\mathrm{Hom}(\pi, G) /\!\!/ G.$$

Roughly: "$\mathcal{R}(\Sigma)$ is the ring of regular functions on $\mathrm{Hom}(\pi, G)/G$."

- α immersed loop on Σ determines a conjugacy class $\bar{\alpha}$ ~~of~~ in π as follows,



let $\alpha_1 = s_1^{-1} * \alpha * s_1$

If s_2 and $\alpha_2 = s_2^{-1} * \alpha * s_2$, then let $\gamma = s_2^{-1} * x * s_1$, we have $\alpha_1 = \gamma^{-1} \cdot \alpha_2 \cdot \gamma$.

$\bar{\alpha}$ is the conjugacy class of α_1

(Goldman - Weil - Pertersson)

Def: $\{,\}: \mathcal{R}(\Sigma) \times \mathcal{R}(\Sigma) \longrightarrow \mathcal{R}(\Sigma)$

$$\{tr_{\bar{\alpha}}, tr_{\bar{\beta}}\} = \sum_{p \in \alpha \cap \beta} \frac{1}{2} \varepsilon(p; \alpha, \beta)\left(tr_{\overline{\alpha_p \beta^{-1}}} - tr_{\overline{\alpha_p \beta}}\right)$$

α, β immersed loops s.t. $\alpha \cap \beta$ are at worst double points.

Thm 1: $(\mathcal{R}(\Sigma), \{,\})$ defines a Poisson structure on $Hom(\pi, G)/G$

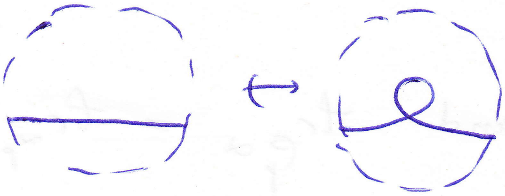
Thm 2: $\mathcal{J}: (\bar{\mathcal{L}}(\Sigma), [,]) \longrightarrow (\mathcal{R}(\Sigma), \{,\})$ by

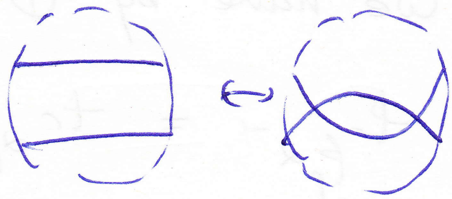
$$\bar{\alpha} \longmapsto -2\, tr_{\bar{\alpha}}$$

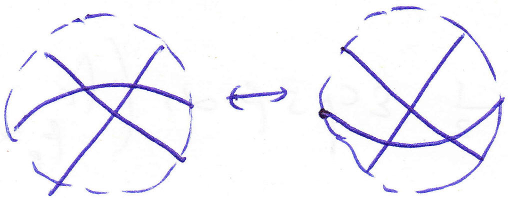
defines a Poisson action.

Pf of thm 1

For the well definition, it suffices to show the invariance under (m1), (m2) and (m3)

(m1): 

(m2) 

(m3): 

- freely homotopic loops determine the same conjugacy class in π_1
- the same argument as yesterday. □

To show $\{ , \}$ is a Poisson bracket, we need

Lemma: For $A, B \in SL(2, \mathbb{C})$

① $\operatorname{tr} A = \operatorname{tr} A^{-1}$,

② $\operatorname{tr} AB = \operatorname{tr} BA$,

③ $\operatorname{tr} A \cdot \operatorname{tr} B = \operatorname{tr} AB + \operatorname{tr} AB^{-1}$.

Anti-symmetry

Since $\beta_p\alpha \sim \alpha_p\beta$ and $\beta_p\alpha^{-1} \sim (\alpha_p\beta^{-1})^{-1}$, we have by ① that

$$tr_{\beta_p\alpha^{-1}} = tr_{\alpha_p\beta^{-1}} \text{ and } tr_{\beta_p\alpha} = tr_{\alpha_p\beta}.$$

Therefore,

$$\{tr_\beta, tr_\alpha\} = \sum_{p\in\beta\cap\alpha} \frac{1}{2}\,\varepsilon(p;\beta,\alpha)\left(tr_{\beta_p\alpha^{-1}} - tr_{\beta_p\alpha}\right)$$

$$= \sum_{p\in\alpha\cap\beta} -\frac{1}{2}\,\varepsilon(p;\alpha,\beta)\left(tr_{\alpha_p\beta^{-1}} - tr_{\alpha_p\beta}\right)$$

$$= -\{tr_\alpha, tr_\beta\}.$$

Jacobi identity.

Letting $p\in\alpha\cap\beta$, $q\in\alpha\cap\gamma$ and $r\in\beta\cap\gamma$, we have

$$\{\{tr_\alpha, tr_\beta\}, tr_\gamma\}$$

$$= \frac{1}{4}\sum_q\sum_p \varepsilon(q;\alpha,\gamma)\,\varepsilon(p;\alpha,\beta)\Big(\underline{\underline{tr_{(\alpha_p\beta^{-1})_q\gamma^{-1}}}} - \underline{\underline{tr_{(\alpha_p\beta^{-1})_q\gamma}}}$$

Ⓘ

$$- tr_{(\alpha_p\beta)_q\gamma^{-1}} + \underline{\underline{tr_{(\alpha_p\beta)_q\gamma}}}\Big)$$

$$+ \frac{1}{4}\sum_r\sum_p \cdots$$

$$\{\{tr_\gamma, tr_\alpha\}\, tr_\beta\}$$

$$= \frac{1}{4}\sum_q \sum_r \cdots$$

$$+ \frac{1}{4}\sum_p \sum_q \varepsilon(p;\alpha,\beta)\,\varepsilon(q;\gamma,\alpha)\Big(-tr_{(\gamma_q\alpha^{-1})_p\beta^{-1}} + tr_{(\gamma_q\alpha^{-1})_p\beta} + tr_{(\gamma_q\alpha)_p\beta^{-1}} - tr_{(\gamma_q\alpha)_p\beta}\Big)$$

II

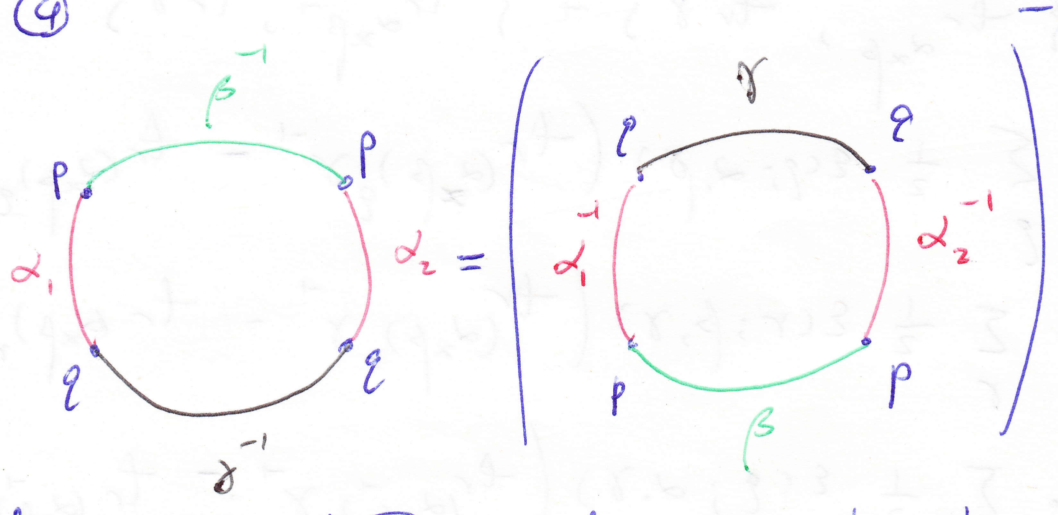
claim: ① $(\alpha_p\beta^{-1})_q\gamma \sim (\gamma_q\alpha)_p\beta^{-1}$

② $(\alpha_p\beta)_q\gamma \sim (\gamma_q\alpha)_p\beta$

③ $(\alpha_p\beta)_q\gamma^{-1} \sim \left((\gamma_q\alpha^{-1})_p\beta^{-1}\right)^{-1}$

④ $(\alpha_p\beta^{-1})_q\gamma^{-1} \sim \left((\gamma_q\alpha^{-1})_p\beta\right)^{-1}$

e.g. ④

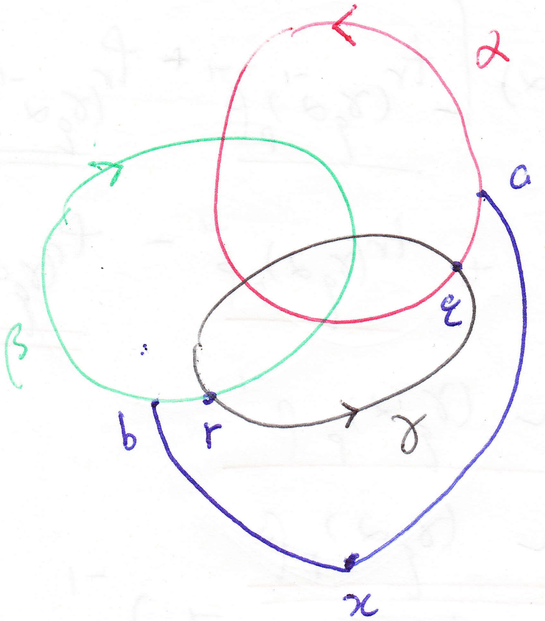


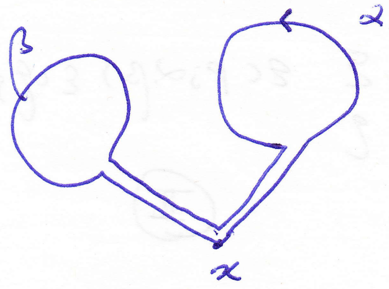
Therefore, I and II cancel out each other.

But similar reason, the other terms cancel out.

Distributivity

$$\{tr_\alpha \cdot tr_\beta , tr_\gamma\} = tr_\alpha \{tr_\beta, tr_\gamma\} + tr_\beta \{tr_\alpha, tr_\beta\}.$$





$\alpha_x \beta$

By ③, $tr_\alpha \cdot tr_\beta = tr_{\alpha_x\beta} + tr_{\alpha_x\beta^{-1}}$.

LHS

$$= \{tr_{\alpha_x\beta}, tr\,\gamma\} + \{tr_{\alpha_x\beta^{-1}}, tr\,\gamma\}$$

$$= \sum_q \tfrac{1}{2}\, \varepsilon(q; \alpha, \beta) \left(tr_{(\alpha_x\beta)_q \gamma^{-1}} - tr_{(\alpha_x\beta)_q \gamma} \right)$$

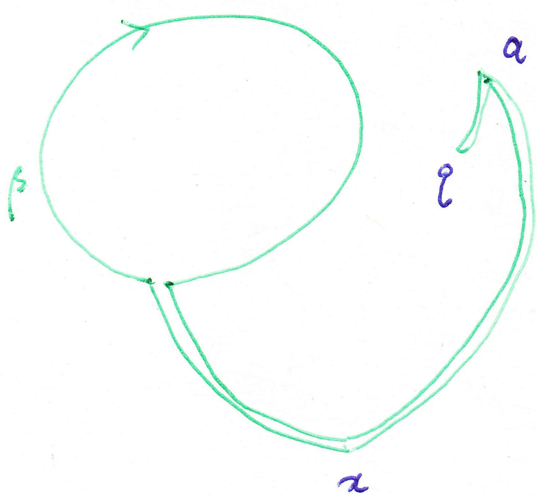
$$+ \sum_r \tfrac{1}{2}\, \varepsilon(r; \beta, \gamma) \left(tr_{(\alpha_x\beta)_r \gamma^{-1}} - tr_{(\alpha_x\beta)_r \gamma} \right)$$

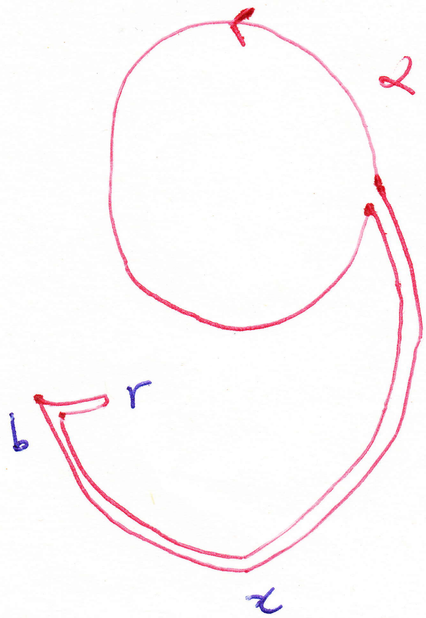
$$+ \sum_q \tfrac{1}{2}\, \varepsilon(q; \alpha, \gamma) \left(tr_{(\alpha_x\beta^{-1})_q \gamma^{-1}} - tr_{(\alpha_x\beta^{-1})_q \gamma} \right)$$

$$+ \sum_r \tfrac{1}{2}\, \varepsilon(r; \beta^{-1}, \gamma) \left(tr_{(\alpha_x\beta^{-1})_r \gamma^{-1}} - tr_{(\alpha_x\beta^{-1})_r \gamma} \right)$$

11

β_{xaq} α_{xbr}





$$= \sum_q \frac{1}{2}\,\varepsilon(q)\left(\mathrm{tr}_{\beta_{xaq}\alpha_q\gamma_q^{-1}} - \mathrm{tr}_{\beta_{xaq}\alpha_q\gamma_q} + \mathrm{tr}_{\beta_{xaq}^{-1}\alpha_q\gamma_q^{-1}} - \mathrm{tr}_{\beta_{xaq}^{-1}\alpha_q\gamma_q}\right)$$

$$+ \sum_r \frac{1}{2}\,\varepsilon(r)\left(\mathrm{tr}_{\alpha_{xbr}\beta_r\gamma_r} - \mathrm{tr}_{\alpha_{xbr}\beta_r\gamma_r} - \mathrm{tr}_{\beta_r^{-1}\alpha_{xbr}\gamma_r^{-1}} + \mathrm{tr}_{\beta_r^{-1}\alpha_{xbr}\gamma_r}\right)$$

$$\overset{2\ 3}{=\!=} \sum_q \frac{1}{2}\,\varepsilon(q;\alpha.\gamma)\,\mathrm{tr}_{\beta_{xaq}}\left(\mathrm{tr}_{\alpha_q\gamma^{-1}} - \mathrm{tr}_{\alpha_q\gamma}\right)$$

$$+ \sum_r \frac{1}{2}\,\varepsilon(r;\beta.\gamma)\,\mathrm{tr}_{\alpha_{xbr}}\left(\mathrm{tr}_{\beta_r\gamma^{-1}} - \mathrm{tr}_{\beta_r\gamma}\right)$$

$$= \mathrm{tr}_\alpha\{\mathrm{tr}_\beta, \mathrm{tr}_\gamma\} + \mathrm{tr}_\beta\{\mathrm{tr}_\alpha, \mathrm{tr}_\gamma\}$$

$\square$

Thm2 is obvious from the formulas.