On Type-Preserving Representations of the Four-Punctured Sphere Group

Tian Yang

Invariants in Low Dimensional Geometry
Gazi Üniversitesi, Ankara, Türkiye

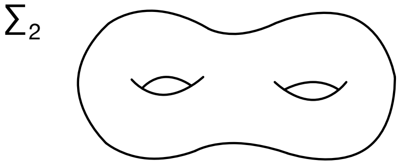
- Goal

  Mapping class group action on the character varieties

- Example

  Proper, ergodic, mixing, etc

- Σ smooth oriented surface, closed, $\chi(\Sigma) < 0$
- Example



- Let $\pi = \pi_1(\Sigma)$, G Lie group, ($G = PSL(2, \mathbb{R})$).

- Representation variety

$$Hom(\pi, G) = \{\text{group homomorphisms } \rho : \pi \to G\}$$

- Remark

  If G algebraic, then $Hom(\pi, G)$ has an affine algebraic variety structure.

- G acts on $Hom(\pi, G)$ by conjugation.

$$(g \circ \rho)(x) = g \cdot \rho(x) \cdot g^{-1}, \qquad \forall\, g \in G,\ x \in \pi.$$

- Character variety

$$X(\Sigma) = Hom(\pi, G)//G$$

- GIT (Geometric Invariant Theory) quotient, space of the closures of orbits.

- Geometric Interpretation

$$X(\Sigma) = \text{space of gauge classes of flat principle G-bundles}$$
$$= \text{deformation space of locally homogeneous structures}$$

- Example

$G = PSL(2, \mathbb{R})$, Teichmüller space $\mathcal{T}(\Sigma) \subset X(\Sigma)$

- Theorem (Goldman, Invent. Math. '88)

$\mathcal{T}(\Sigma)$ is a connected component of $X(\Sigma)$, consisting of Fuchsian (discrete and faithful) representations.

- Mapping class group

$$Mod(\Sigma) = Diff^+(\Sigma)/Diff_0^+(\Sigma)$$

- $Diff^+(\Sigma) = \{$orientation pres. self-diffeomorphisms of $\Sigma\}$

  $Diff_0^+(\Sigma) = \{\phi \in Diff^+(\Sigma) \mid \phi \simeq id_\Sigma\}$

  "$\simeq$" homotopy, or equivalently isotopy.

- Remark

  $Mod(\Sigma)$ is a discrete group.

- Dehn-Nielsen Theorem

$$Mod(\Sigma) \cong Out(\pi)$$

- $Out(\pi) \doteq Aut(\pi)/Inn(\pi)$

  $Aut(\pi) = \{\text{automorphisms of } \pi\}$

  $Inn(\pi) = \{\text{conjugations by elements of } \pi\}$

- $Aut(\pi)$-action on $Hom(\pi, G)$

  $\Longrightarrow$ $Out(\pi)$-action on $X(\Sigma)$

  $\Longrightarrow$ $Mod(\Sigma)$-action on $X(\Sigma)$

- Theorem (Goldman, Ann. Math. '97)

  $G = SU(2)$, $Mod(\Sigma)$-action on $X(\Sigma)$ is ergodic.

- $(X, \mathcal{B}, m)$ measure space. A measure preserving G-action is ergodic if each G-invariant subset is either of measure 0 or of full measure.

- Theorem (Goldman, Adv. Math. '84)

  There is a $Mod(\Sigma)$-invariant symplectic structure on $X(\Sigma)$, inducing a $Mod(\Sigma)$-invariant measure.

- Theorem (Pickrell-Xia, Comment. Math. Helv. '02)

  G compact, $Mod(\Sigma)$-action on each connected component of $X(\Sigma)$ is ergodic.

- Question

  What about G non-compact? $G = PSL(2, \mathbb{R})$.

- Theorem (Goldman, Invent. Math. '88)

  $X(\Sigma_g)$ has $4g - 3$ connected components, indexed by the Euler class, two of which are Teichmüller spaces $\mathcal{T}(\Sigma)$ and $\mathcal{T}(\Sigma^{op})$.

- Milnor-Wood Inequality

$$2 - 2g \leqslant e(\rho) \leqslant 2g - 2$$

- "$=$" holds $\iff$ Fuchsian $\iff$ Teichmüller

- Theorem (Fricke, 1897)

  $Mod(\Sigma)$-action on $\mathcal{T}(\Sigma)$ is properly discontinuous.

- Question

  What about the other components?

- Conjecture (Goldman)

  $Mod(\Sigma)$-action on each non-Teichmüller component is ergodic.

- Question

  What is known?

- Answers

  (Marché-Wolff, '13) True for Σ_2.

  (Souto, '14) True for Euler class 0 component for any Σ_g.

- Theorem (Marché-Wolff, '13)

  Bowditch's Conjecture $\iff$ Goldman's Conjecture

- Conjecture (Bowditch, Proc. London Math. Soc. '98)

  Every non-Fuchsian representation sends some simple loop to a non-hyperbolic element of $PSL(2,\mathbb{R})$.

- Classification of $A \in PSL(2,\mathbb{R})$

$$\begin{aligned} \text{elliptic} &\iff |trA| < 2 \\ \text{parabolic} &\iff |trA| = 2 \\ \text{hyperbolic} &\iff |trA| > 2 \end{aligned}$$

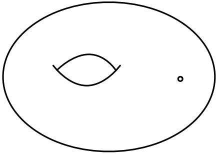
- Remark

  Bowditch's Conjecture is originally for type-preserving representations of punctured surface groups.

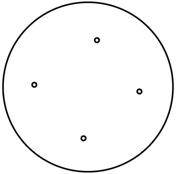
- Σ smooth oriented punctured surface, $\chi(\Sigma) < 0$

- Example

$\Sigma_{1,1}$

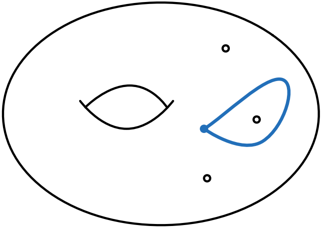


$\Sigma_{0,4}$



- Type-preserving representation

  $\rho : \pi \to PSL(2, \mathbb{R})$ sending each peripheral element to a parabolic element

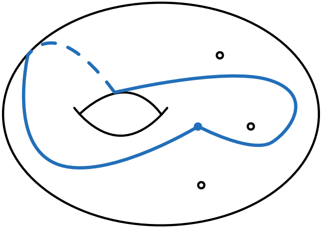


- Geometrically,

$$\text{punctures} \longmapsto \text{cusps}$$

- Conjecture (Bowditch, Proc. London Math. Soc. '98)

  Every non-Fuchsian type-preserving $\rho : \pi \to PSL(2, \mathbb{R})$ sends some non-peripheral simple loop to a non-hyperbolic element.



- True (trivially) for $\Sigma_{0,3}$ and $\Sigma_{1,1}$.

- Main Results (Y. 2014)

  Theorem 1 There are uncountably many type-preserving $\rho : \pi_1(\Sigma_{0,4}) \to PSL(2,\mathbb{R})$ with $e(\rho) = \pm 1$ sending each non-peripheral simple loop to a hyperbolic element.

  Theorem 2 Every type-preserving $\rho : \pi_1(\Sigma_{0,4}) \to PSL(2,\mathbb{R})$ with $e(\rho) = 0$ sends some non-peripheral simple loop to a non-hyperbolic element.

  Theorem 3 $Mod(\Sigma_{0,4})$-action on each non-Teichmüller component of $X(\Sigma_{0,4})$ is ergodic.

- Lengths coordinate

$\mathcal{T}$ ideal triangulation of Σ, E edges, T triangles

Theorem (Kashaev, Math. Res. Lett. '05)

$$(\lambda, \epsilon) \in \mathbb{R}^{E}_{>0} \times \{\pm 1\}^{T} \Longrightarrow \rho : \pi \to PSL(2, \mathbb{R})$$

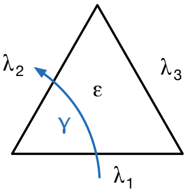
with

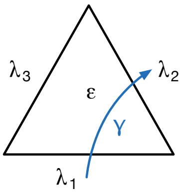
$$e(\rho) = \frac{1}{2} \sum_{t \in T} \epsilon(t).$$

- Remark

When all $\epsilon(t) = 1$, Penner's lengths coordinate for the decorated Teichmüller space.

- Trace formula



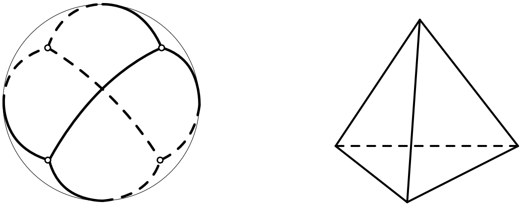


$$M = \begin{bmatrix} \lambda_1 & \epsilon\lambda_3 \\ 0 & \lambda_2 \end{bmatrix} \qquad M = \begin{bmatrix} \lambda_2 & 0 \\ \epsilon\lambda_3 & \lambda_1 \end{bmatrix}$$

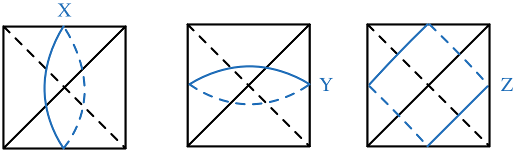
Theorem (Kashaev, Math. Res. Lett. '05)

$$|tr\rho(\gamma)| = \left| \frac{tr\, M_1 \cdots M_n}{\lambda_1 \cdots \lambda_n} \right|$$

- Tetrahedral triangulations



- Distinguished simple loops



- Traces of distinguished loops

  For $(\lambda, \epsilon) = (a, b, c, d, e, f, -1, 1, 1, 1)$, let

$$x = ab, \quad y = cd \quad z = ef.$$

  Then $e(\rho) = 1$, and

$$|tr\rho(X)| = \left|\frac{y^2 + z^2 - x^2}{yz}\right|$$
$$|tr\rho(Y)| = \left|\frac{x^2 + z^2 - y^2}{xz}\right|$$
$$|tr\rho(Z)| = \left|\frac{x^2 + y^2 - z^2}{xy}\right|.$$

- Cosine Law of Euclidean triangles.

- For (x, y, z) satisfying one of

$$\begin{aligned} x &> y + z \\ y &> x + z \\ z &> x + y \end{aligned} \qquad i.e., \qquad (x, y, z) \in$$  ,

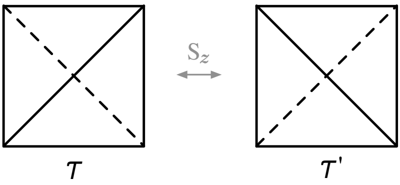
one has $|tr\rho(X)| > 2$, $|tr\rho(Y)| > 2$, $|tr\rho(Z)| > 2$.

- Distinguished simple loops are hyperbolic.

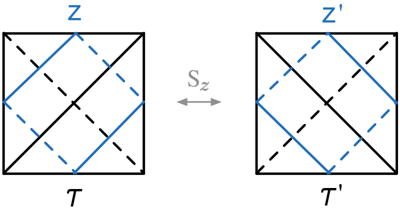
- What about other simple loops?

  Every simple loop is distinguished in some tetrahedral triangulation.

- Simultaneous diagonal switches



- Change of distinguished loops



- Change of lengths

  If $\lambda' = (a', b', c', d', e', f')$, and

  $$x' = a'b', \quad y' = c'd', \quad z' = e'f',$$

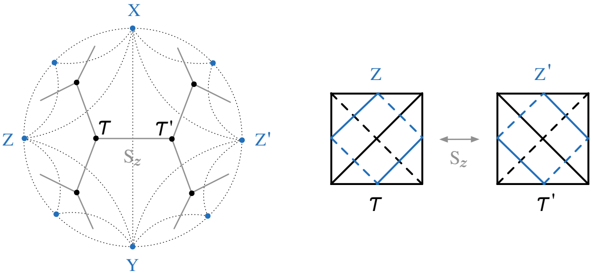
  then $x' = x$, $y' = y$ and

  $$z' = \left|\frac{x^2 - y^2}{z}\right|.$$

- Inequalities are preserved

  $(x, y, z) \in$  $\iff$ $(x', y', z') \in$ 

- Relationship with Farey diagram



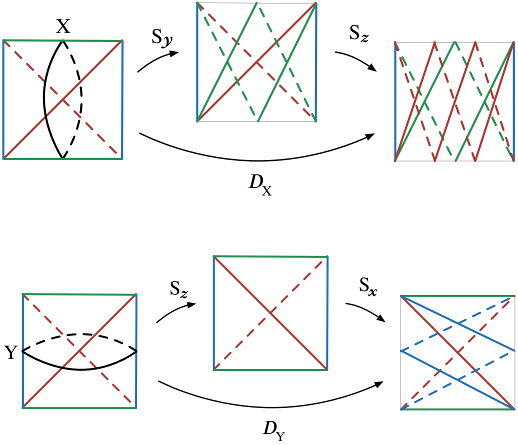
- Any two tetrahedral triangulations are related by simultaneous diagonal switches.

- All simple loops are hyperbolic, and ρ is non-Fuchshian.

- Counterexamples form full measured subset of . $\square$

- $Mod(\Sigma_{0,4}) \cong F_2$ generated by Dehn twists D_X and D_Y.

- Dehn twists and simultaneous diagonal switches



- $Mod(\Sigma_{0,4})$-action on 

$$D_X : \ (x,y,z) \ \mapsto \ \left(x, \left| \frac{x^2 - z^2}{y} \right|, \left| \frac{\left(\frac{x^2-z^2}{y}\right)^2 - x^2}{z} \right| \right)$$

$$D_Y : \ (x,y,z) \ \mapsto \ \left(\left| \frac{y^2 - \left(\frac{x^2-y^2}{z}\right)^2}{x} \right|, y, \left| \frac{x^2 - y^2}{z} \right| \right)$$

- Recall

$$F_2 \cong \Gamma(2)/\pm I,$$

where

$$\Gamma(2) = \left\{ A \in SL(2,\mathbb{Z}) \;\middle|\; A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2 \right\}.$$

- Group homomorphism

$$\phi : \Gamma(2) \to Mod(\Sigma_{0,4})$$

given by

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \mapsto D_X \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mapsto D_Y.$$

- Consider ϕ-equivariant two-fold covering

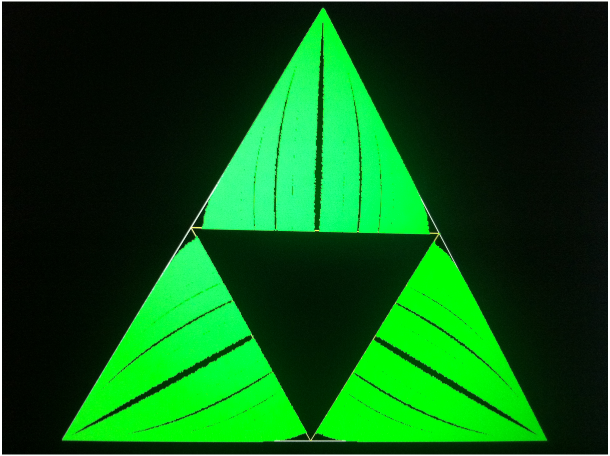
$$\psi : \mathbb{R}^2 \rightarrow$$ 

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} |\sinh s| \\ |\sinh t| \\ |\sinh(s+t)| \end{pmatrix}.$$

- Moore's Theorem

  $\Gamma(2)$-action on $\mathbb{R}^2$ is ergodic.

- $Mod(\Sigma_{0,4})$-action on  is ergodic. □



- Traces of distinguished loops

Let $(\lambda, \epsilon) = (a, b, c, d, e, f, -1, -1, 1, 1)$, and

$$x = ab, \quad y = cd \quad z = ef.$$

Then $e(\rho) = 0$, and

$$|tr\rho(X)| = \left|\frac{x^2 + y^2 + z^2 - 2xy - 2xz}{yz}\right|$$
$$|tr\rho(Y)| = \left|\frac{x^2 + y^2 + z^2 + 2yz - 2xy}{xz}\right|$$
$$|tr\rho(Z)| = \left|\frac{x^2 + y^2 + z^2 + 2yz - 2xz}{xy}\right|.$$

- Change of lengths

If $\lambda' = (a', b', c', d', e', f')$ and

$$x' = a'b', \quad y' = c'd', \quad z' = e'f',$$

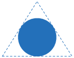
then

$$x' = \frac{(y+z)^2}{x},$$

$$y' = \frac{(x-z)^2}{y},$$

$$z' = \frac{(x-y)^2}{z}.$$

- If (x, y, z) satisfy one of

$$\begin{aligned} \sqrt{x} &\leqslant \sqrt{y} + \sqrt{z} \\ \sqrt{y} &\leqslant \sqrt{x} + \sqrt{z} \\ \sqrt{z} &\leqslant \sqrt{x} + \sqrt{y} \end{aligned} \qquad i.e., \qquad (x, y, z) \in$$  ,

then $|tr\rho(X)| \leqslant 2$.

- <u>Trace reduction algorithm</u>

  Doing simultaneous diagonal switches along the longest pair of opposite edges.

- Trace reduction algorithm stops in finitely many steps. □

— THANK YOU —