Hyerbolic Cone Metrics on 3-manifolds with Boundary

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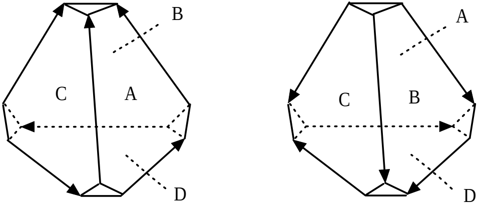
- Preliminaries

  M 3-manifold, $\partial M \neq \emptyset$ and $\chi(\partial M) < 0$

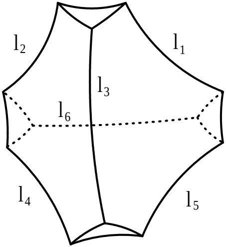
  $\mathcal{T}$ ideal triangulation of M

- Example



- Hyperbolic cone metric

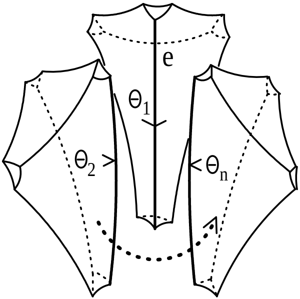
  a metric on M that restricts to hyperideal tetrahedra to the tetrahedra in $\mathcal{T}$



  $\mathcal{L}(M,\mathcal{T})$ moduli space of hyperbolic cone metrics

  $\mathcal{L}(M,\mathcal{T}) \subset \mathbb{R}^E_{>0}$ parametrized by the edge lengths.

- Cone angles



sum of dihedral angles

- Combinatorial curvature

$K : \mathcal{L}(M, \mathcal{T}) \to \mathbb{R}^E$ defined by

$$K(l)(e) = 2\pi - \text{cone angle at } e$$

- Questions

1. (Global Rigidity) Do combinatorial curvatures determine the hyperbolic cone metrics?

   I.e., is $K : \mathcal{L}(M, \mathcal{T}) \to \mathbb{R}^E$ injective?

2. Describe the space $Im(K)$ of combinatorial curvatures.

   $0 \in Im(K) \Longrightarrow$ existence of hyperbolic metric

- Main results

  1. Theorem 1 (Luo -Y.)
     The map $K : \mathcal{L}(M, \mathcal{T}) \to \mathbb{R}^E$ is a smooth embedding.
     In particular, hyperbolic cone manifolds are globally rigid.

  2. Theorem 2 (Luo -Y.)
     $Im(K) \cap (\pi, 2\pi)^E$ is a concrete convex open polytope in $\mathbb{R}^E$.

- Similar results hold for M with torus or mixed boundary.

- Related results

  Hodgson-Kerckhoff, Weiss

  Mazzeo-Montcouquiol, Montcouquiol

  Andreev, Hodgson-Rivin, Rivin, Bao-Bonahon

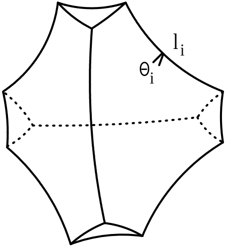
- Variational principle

1. If $X \subset \mathbb{R}^n$ open convex and $f : X \to \mathbb{R}$ smooth strictly convex, then $\nabla f : X \to \mathbb{R}^n$ is injective.

2. If the Hessian $H(f)$ positive definite for all $x \in X$, then ∇f is a smooth embedding.

- Schlaefli formula

$$\frac{\partial Vol}{\partial \theta_i} = -\frac{l_i}{2}$$



- Thoerem (Schlenker)

  Vol is smooth strictly concave in θ_i.

- Legendre transformation

$$W(l_1, \ldots, l_6) = 2Vol + \sum_{i=1}^{6} \theta_i l_i$$

Up to a constant,

$$W(l) = \int^{l} \sum_{i} \theta_i dl_i.$$

- Proposition

W is smooth strictly convex, and

$$\frac{\partial W}{\partial l_i} = \theta_i.$$

- Co-volume function

  Consider $W : \mathcal{L}(M, \mathcal{T}) \to \mathbb{R}$ defined by

$$W(l) = \sum_{\sigma \in \mathcal{T}} W(l_\sigma) - 2\pi \sum_{e \in E} l_e.$$

  Then W is smooth strictly convex, and

$$\nabla W = -K.$$

- $\mathcal{L}(M, \mathcal{T})$ is not convex !

- Theorem (Luo)

  The map $K : \mathcal{L}(M, \mathcal{T}) \to \mathbb{R}^E$ is a local embedding.

- Key Lemma

  W can be extended to a C^1-smooth convex function

  $$\widetilde{W} : \mathbb{R}^E_{>0} \to \mathbb{R}.$$

- Proof of Thm 1

  Suppose existed $l_1 \neq l_2 \in \mathcal{L}(M, \mathcal{T})$ with $K(l_1) = K(l_2)$.

  Let $L \subset \mathbb{R}^E_{>0}$ be the line segment in jointing l_i.

  $K(l_1) = K(l_2) \implies \widetilde{W}|_L$ is affine.

  $\widetilde{W}|_{\mathcal{L}(M,\mathcal{T})} = W \implies \widetilde{W}|_L$ is strictly convex near l_i.

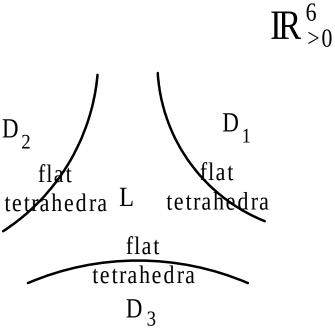
- The map $K : \mathcal{L}(M, \mathcal{T}) \to \mathbb{R}^E$ is injective. $\square$

- Degeneration of hyperideal tetrahedra

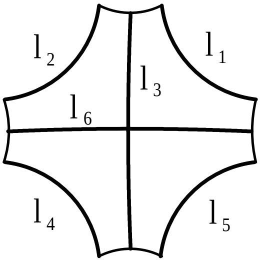
  $\mathcal{L}$ space of hyperideal tetrahedra

  $\mathcal{L} \subset \mathbb{R}^6_{>0}$ parametrized by the edge lengths.

- Proposition

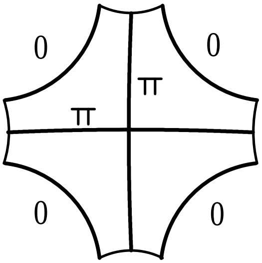


- Degeneration of hyperideal tetrahedra



flat tetrahedra

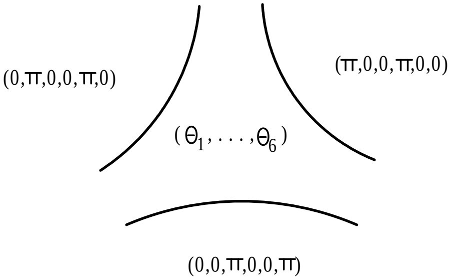
- Degeneration of hyperideal tetrahedra



dihedral angles

- Extended dihedral angles

$$\widetilde{\theta}_i : \mathbb{R}^6_{>0} \to \mathbb{R}$$



- Lemma

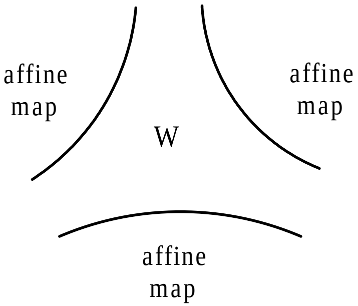
$\widetilde{\theta}_i : \mathbb{R}^6_{>0} \to \mathbb{R}$ is continuous.

- Proposition

The function $\widetilde{W} : \mathbb{R}^6_{>0} \to \mathbb{R}$ defined by

$$\widetilde{W}(l) = \int^{l} \sum_i \widetilde{\theta}_i dl_i$$

is C^1-smooth convex, and $\widetilde{W}|_{\mathcal{L}} = W$.



- Extended co-volume function

  The function $\widetilde{W} : \mathbb{R}^E_{>0} \to \mathbb{R}$ defined by

$$\widetilde{W}(l) = \sum_{\sigma \in \mathcal{T}} \widetilde{W}(l_\sigma) - 2\pi \sum_{e \in E} l_e$$

  is C^1-smooth convex, and $\widetilde{W}|_{\mathcal{L}(M,\mathcal{T})} = W$.

- Other results

  Relationship with angle structures

  an assignment of dihedral angles to $(M, \mathcal{T})$ so that

  1. each tetrahedron is hyperideal,
  2. the cone angle at each edge is 2π.

- Theorem (Casson, Lackenby, Rivin)

  "maximum volumed angle structure $\Longrightarrow$ hyperbolic metric"

$$\text{maximum volume} \; = \; \text{hyperbolic volume}.$$

- Other results

  Maximum volumed angle structure doesn't always exist!

- Conjecture (Casson)

  For a triangulated hyperbolic 3-manifold,

$$\text{supremum of volumes} \leqslant \text{hyperbolic volume.}$$

- Other results

  semi-angle structures

  an assignment of dihedral angles to $(M, \mathcal{T})$ so that

  1. each tetrahedron is hyperideal or flat,
  2. the cone angle at each edge is 2π.

- Theorem 3 (Luo -Y.)

  The maximum volumed semi-angle structure is the extended dihedral angles of some $l \in \mathbb{R}^E_{>0}$.

  "maximum volumed semi-angle structure $\Longrightarrow$ extended hyperbolic metric"

THANK YOU !