Lecture 16: $TV = |RT|^2$.

Thm (Turaev, Walker, Roberts).

M closed, orientable 3-mfd. Then $\forall r \geq 3$,

$$\boxed{TV_r(M) = |I_r(M)|^2.}$$

Recall: (M, T). V, E, F, T = sets of vertices, edges, faces, tetrahedra. $\eta = \dfrac{-2r}{(A^2 - A^{-2})^2}$.

$$\boxed{TV_r(M) = \eta^{-|V|} \sum_{c \in A_r} \prod_{e \in E} \bigcirc \prod_{f \in F} \ominus^{-1} \prod_{\sigma \in T} \circledast}$$

$M = M_L$, $D = D(L)$ standard, σ = signature of linking matrix. $\mu = \dfrac{A^2 - A^{-2}}{\sqrt{-2r}}$. $(\eta = \mu^{-2})$.

$$\boxed{I_r(M) = \mu \langle \mu\omega_r, \ldots, \mu\omega_r \rangle_D \langle \mu\omega \rangle_{u_+}^{-\sigma}}$$

idea:

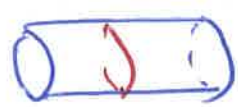
$$\begin{array}{ccccc}
\text{triangulation of } M & \longrightarrow & \text{Heegaard splitting of } M & \hookleftarrow & \text{Kirby diagram of } M\#(-M) \\
\downarrow & & \downarrow & & \downarrow \\
TV & = & \text{Roberts' invariant} & = & I_r(M\#(-M)).
\end{array}$$

Handlebody:

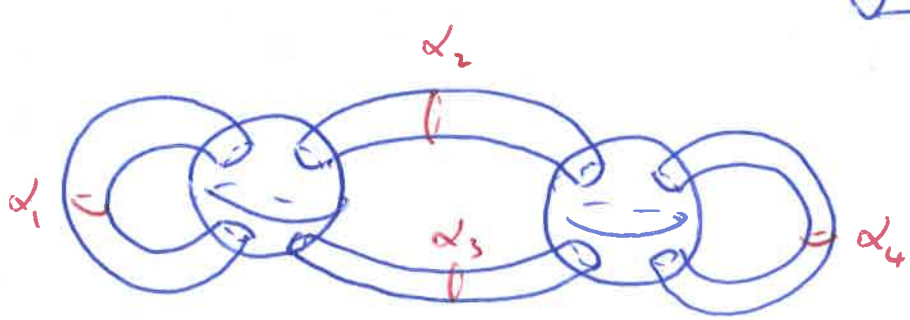
H_g.

$$H_g = \{0\text{-handles}\} \cup \{1\text{-handles}\}$$

$$\{0\text{-handles}\} = D^3 \times \{0\}, \qquad \{1\text{-handles}\} = D^2 \times D^1.$$

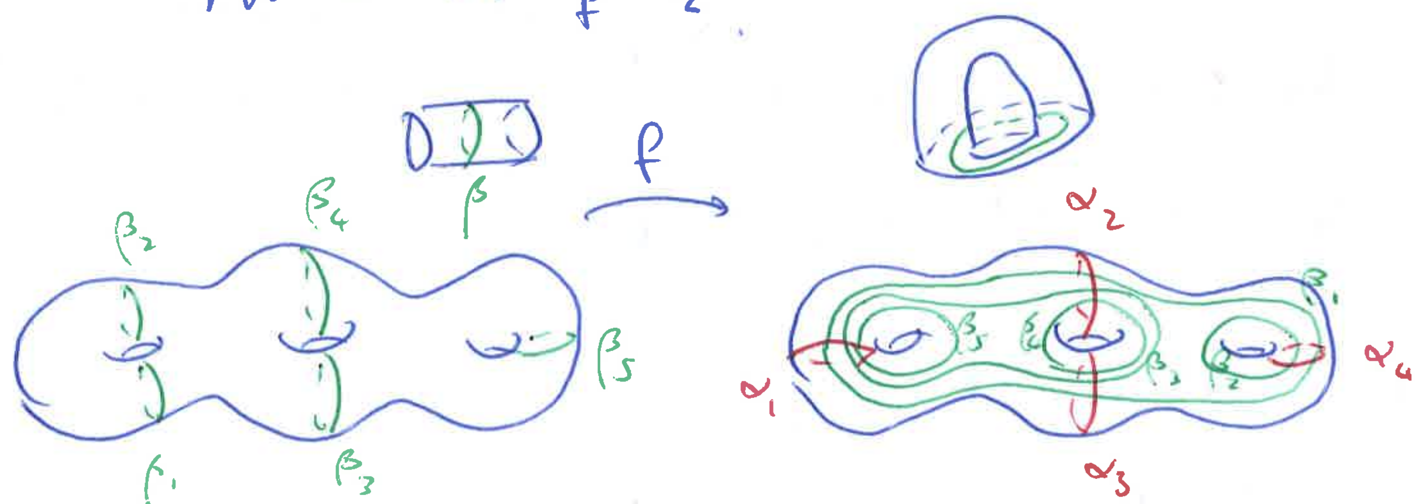


$\alpha = \partial D^2 \times \{\tfrac{1}{2}\}$.



Def: A Heegaard splitting of a closed 3-mfd M consists of two handlebodies $H_1 \cong H_2 \cong H_g$ and a homeomorphism $f: \partial H_2 \to \partial H_1$ s.t.

$$M \cong H_1 \cup_f H_2$$



- f is determined by image of β-curves

$\Rightarrow$ handle structure of M

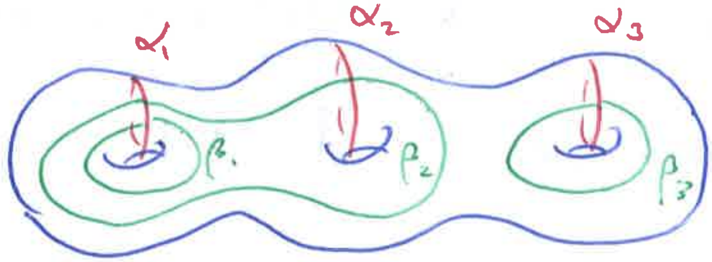
$$H_1 = \{0\text{-handles}\} \cup \{1\text{-handles}\}$$

$$H_2 = \{2\text{-handles}\} \cup \{3\text{-handles}\}$$

$$d_i = \#\text{ of } i\text{-handles}$$

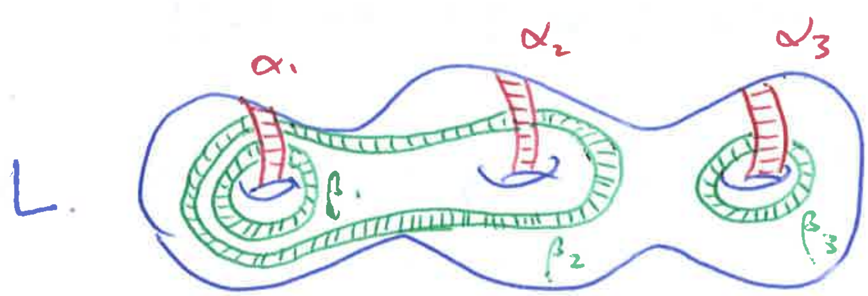
$$\left(\begin{array}{l} d_1 = \#\text{ of } \alpha\text{-curves} \\ d_2 = \#\text{ of } \beta\text{-curves} \end{array} \right)$$

- Heegaard diagram (S, α, β)



- Roberts' invariant (chain-mail) is constructed by the following steps.

1) embed H_1 in S^3,
2) thicken α- and β-curves along ∂H_1,
3) push β-curves slightly into H_1 to get a framed link L in S^3.



4) Definition:

$$CH_r(m) = \mu^{d_0+d_3} \langle \mu\omega_r, \cdots, \mu\omega_r \rangle_L$$

Thm (Roberts)

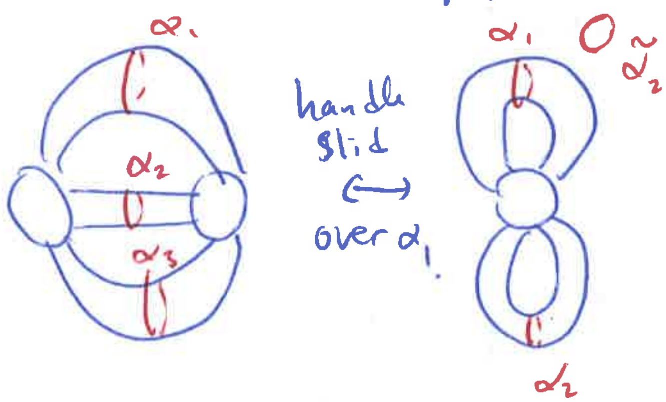
1) $CH_r(M)$ defines an invariant of M, ie, is independent of the Heegaard splitting and the embedding of H_1.

2) $CH_r(M) = TV_r(M)$.

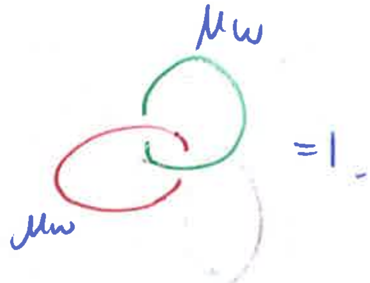
3) $CH_r(M) = |I_r(M)|^2$.

pf of 1): Any two H.S. are differed by 0-1, 1-2, 2-3 birth and dies, 1- and 2-handle slids.

0-1 (2-3 by duality)



1-2



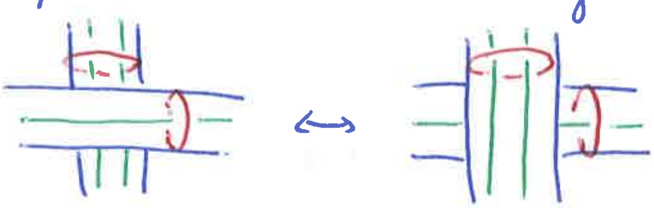
1-handle slide doesn't change diagram

2-handle slide

$\updownarrow$

KM II.

Any two embeddings of H_1 are differed by



and twist

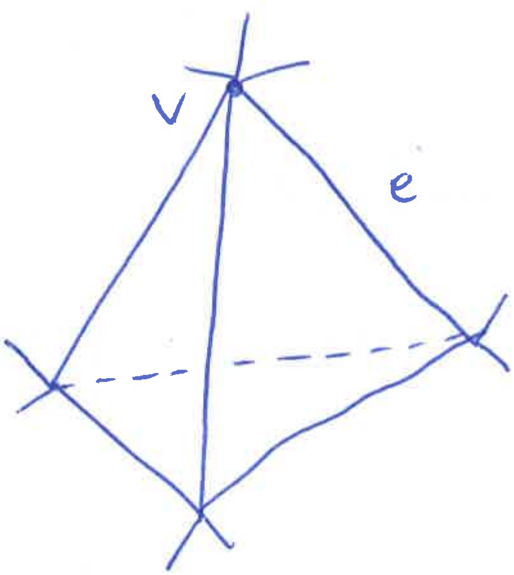
which are composition of KM I and KM II's.

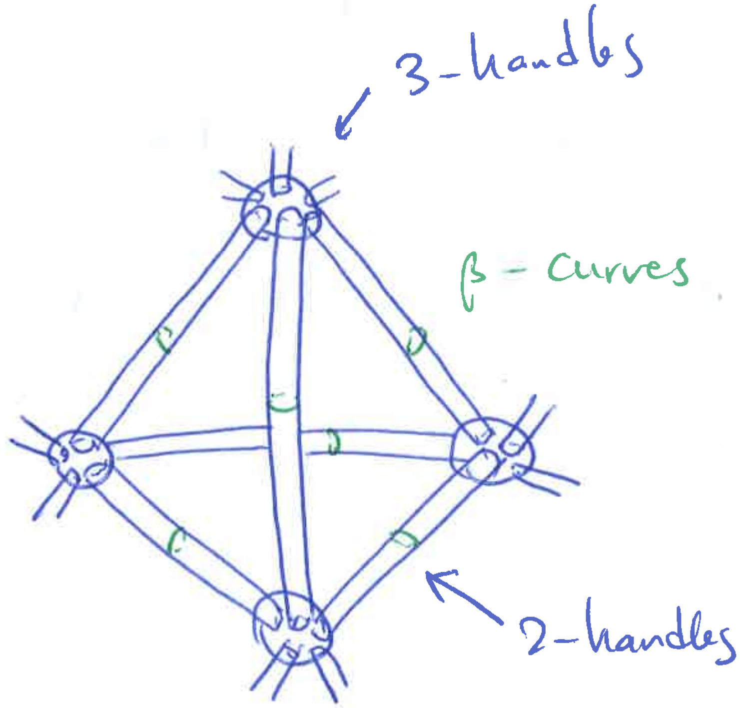
For proof of 2), we need

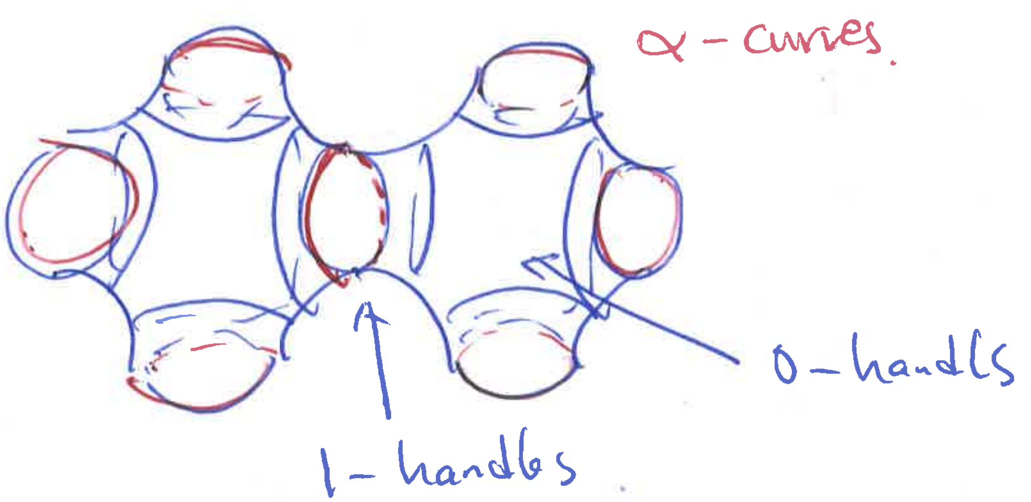
<u>Heegaard splitting from a triangulation</u> $\mathcal{T}$.

H_2 = tubular nbhd of 1-skeleton $E \cup V$.

$H_1 = M \setminus H_2$.



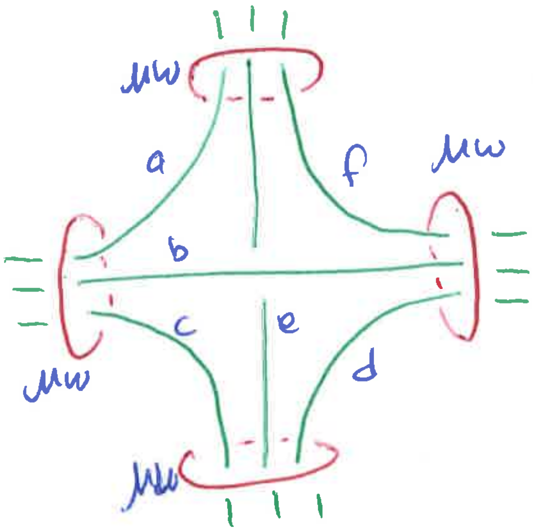




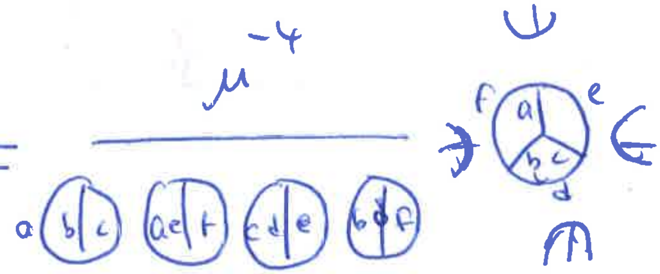
~~$d_0 = |T|$~~

$d_0 = |T|$, $d_1 = |F|$, $d_2 = |E|$, $d_3 = |V|$.

- $L_g = \{\alpha\text{-curves}\} \cup \{\beta\text{-curves}\}$ has the property that every α-curve encloses 3 β-curves.



Fushion Rule



Then $CH_r(m) = \mu^{d_0+d_3} \langle \mu\omega, \dots, \mu\omega \rangle_{L_g}$

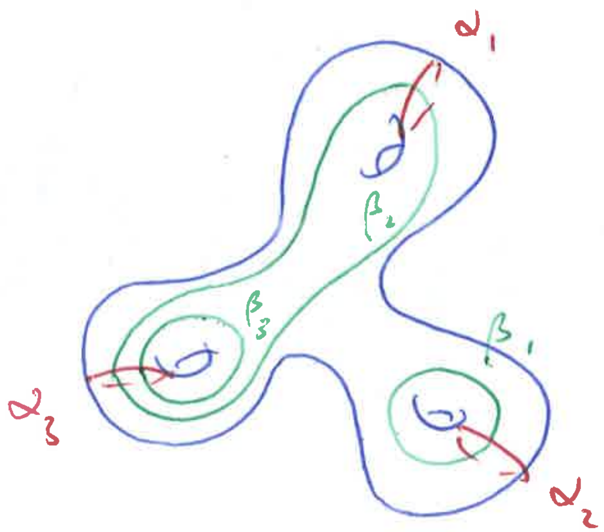
$$= \mu^{d_0+d_3+d_2-d_1} \sum_{c \in \mathcal{A}_r} \prod_e \bigcirc \prod_f \ominus^{-1} \prod_\sigma \circledast$$

$$\left(\begin{array}{l} d_0 - d_1 + d_2 - d_3 = 0 \Rightarrow d_0 + d_3 + d_2 - d_1 = 2d_3 = 2|V| \\ \mu^{d_0+d_3+d_2-d_1} = \mu^{2|V|} = \eta^{-|V|} \end{array} \right)$$

$$= \eta^{-|V|} \sum_{c \in \mathcal{A}_r} \prod_e \bigcirc \prod_f \ominus^{-1} \prod_\sigma \circledast = TV_r(m).$$

Standard Heegaard splitting.

(H_1, H_2, f) s.t both H_1 and H_2 have exactly one 0-handle.



L_s

~~Then~~ Prop: $\mathcal{M}_{L_s} = \mathcal{M} \# (-\mathcal{M})$.

Then $CH_r(\mathcal{M}) = \mu^2 \langle \mu\omega, \dots, \mu\omega \rangle_{L_s}$

$= \mu^2 \cdot \mu^{-1} \cdot I_r(\mathcal{M} \# (-\mathcal{M})) \cdot \langle \mu\omega \rangle_{u_+}^{\sigma(L_s)}$

$= \mu^2 \cdot \mu^{-1} \cdot \mu^{-1} |I_r(\mathcal{M})|^2 \langle \mu\omega \rangle_{u_+}^{\sigma(L_s)}$

$\overset{\text{Lemma below}}{=} |I_r(\mathcal{M})|^2$.

Lemma: $\sigma(L_s) = 0$.

$\square$

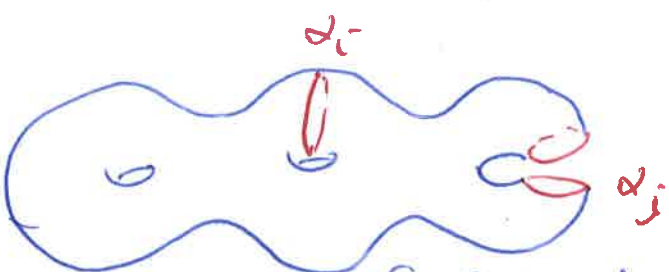
Pf of Prop: By definition, $M_{L_s} = \partial X_{L_s}$, where X_{L_s} is 4-mfd from B^4 by attaching 2-handles along L_s. Let X'_{L_s} be 4-mfd from B^4 by attaching 1-handles to α-curves and 2-handles to β-curves. Then

(i) $\partial X_{L_s} = \partial X'_{L_s}$,

(ii) $X'_{L_s} = M^{(2)} \times I$, where $M^{(2)} = M \setminus B^3$
$= 2\text{-skeleton of } M$

(iii) $\partial X'_{L_s} = M \# (-M)$ $\square$

Pf of Lemma: The linking matrix $Lk(L_s)$ has the form $\begin{array}{cc} \left(\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array}\right) & \begin{array}{l} \}\alpha \\ \}\beta \end{array} \end{array}$, since $lk(\alpha_i, \alpha_j) = lk(\beta_i, \beta_j) = 0$



S_j is the Seifert surface.

- Signature of such matrix ~~has st~~ equals 0 since if $v = (v_1, v_2)$ is eigenvector of e.v. λ, then $(-v_1, v_2)$ is eigenvector of e.v $-\lambda$ $\square$