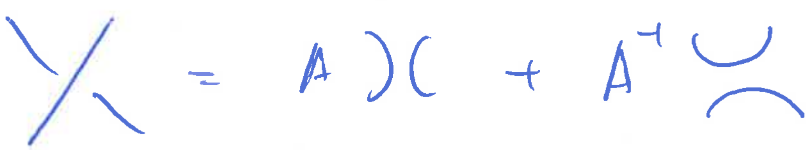
Lecture 12: Kauffman bracket skein modules

- M compact oriented 3-mfd (not necessarily closed). The Kauffman bracket skein module $K_A(M)$ of M is the $\mathbb{C}$-module generated by isotopy classes of framed links in M modulo the relations

(1) KB skein rel'n: $A \in \mathbb{C} \setminus \{0\}$,



(2) Framing rel'n:

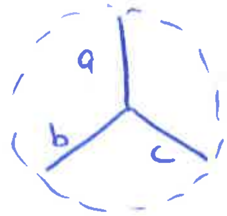
$$\bigcirc \sqcup L = (-A^2 - A^{-2}) L.$$

Eg: $K_A(S^3) \cong \mathbb{C}$

$$L \longmapsto \langle L \rangle$$ ← Kauffman bracket.

Prop: $A = e^{\frac{\pi i}{2r}}$. Then in $K_A(S^3)$, any skein containing

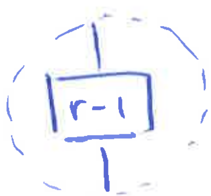
or



where (a,b,c) non r-admissible vanish.

Def: let $A = e^{\frac{\pi i}{2r}}$. The reduced skein module $K_A^{red}(M)$ of M is the quotient of $K_A(M)$ by the relations (3), (4).

(3)



$= 0$,

(4)

$= 0$, for (a,b,c) non r-admissible.

Eg: $K_A^{red}(S^2 \times S^1) \cong \mathbb{C}$.

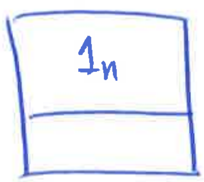
$S^2 \times S^1 = D(H) =$ 

$i: H \hookrightarrow S^2 \times S^1$ induces

$i: K_A(H) \longrightarrow K_A(S^2 \times S^1)$ and

$i: K_A^{red}(H) \longrightarrow K_A^{red}(S^2 \times S^1)$

$e_n \longmapsto e_n$

Recall: 

$= \sum_k c_k f_k$

$\uparrow$ Jones-Wenzl.

let $L \subset S^2 \times S^1$. Let $S^2 = S^2 \times \{pt\} \subset S^2 \times S^1$

and $S^1 = \{pt\} \times S^1 \subset S^2 \times S^1$.

Can isotope L s.t.

$L \cap S^1 = \phi$ and

$|L \cap S^2| =$ minimum.

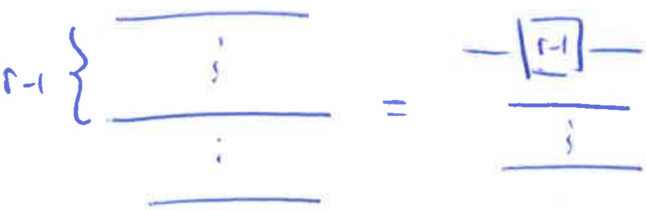
Then $L \subset i(H) \left(= S^2 \setminus S^1\right)$.

Recall $1_n = \sum_k c_k f_k$.

$L = \sum_n d_n e_n \in K_A(S^2 \times S^1)$.

$\uparrow$ k-th Chebyshev poly

Case 1: $n \geq r-1$.



$+ L' = L' \in K_A^{red}(S^2 \times S^1)$,

where L' has fewer intersections w/ S^2 than L.

By induction, it reduces to

Case 2. $n \leq r-2$.

If $n \neq 0$, then $e_n = 0 \in K_A^{red}(S^2 \times S^1)$, since

$(-A^2 - A^{-2})\, e_n =$  $=$  $= (-A^{2n+2} - A^{-2n-2})\, e_n$.

and $(-A^2 - A^{-2}) \neq (-A^{2n+2} - A^{-2n-2})$.

$L = d_0 e_0 \in K_A^{red}(S^2 \times S^1) \cong \mathbb{C}$

$L \mapsto d_0$

Using the same trick, one can prove

Thm: $K_A^{red}(M \# N) \cong K_A^{red}(M) \otimes K_A^{red}(N)$

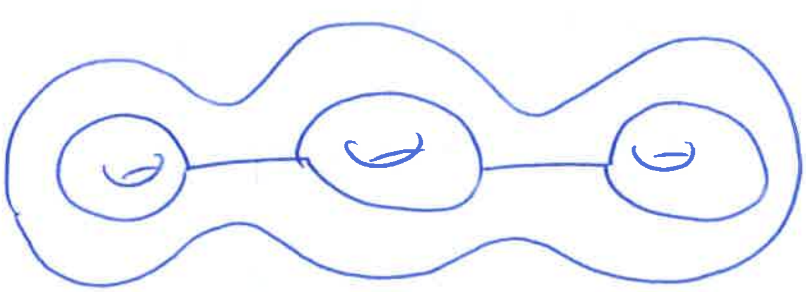
Cor: $K_A^{red}(\#_k S^2 \times S^1) \cong \mathbb{C}$.

Eg: $K_A^{red}(H_g)$, where H_g = 

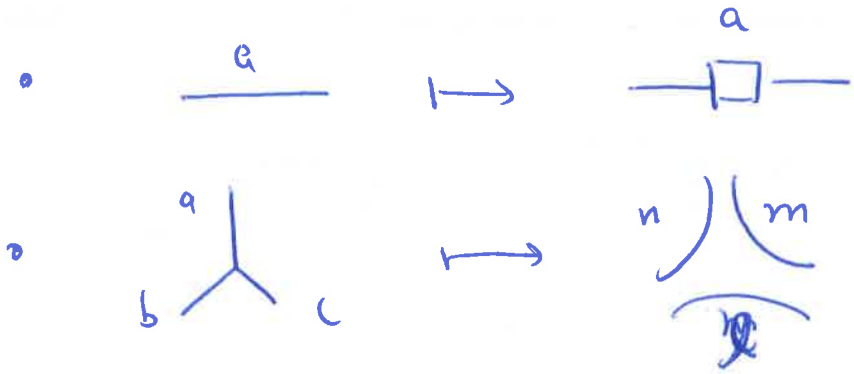
handle body of genus g.

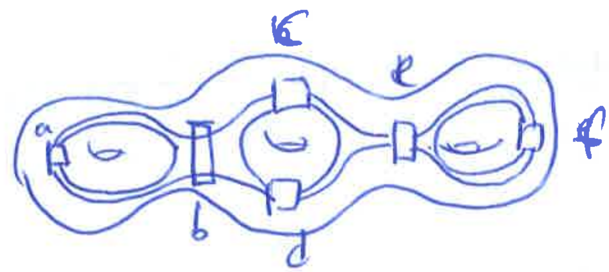
Rm. This is the vector space of Reshetikhin-Turaev TQFT for $\Sigma_g = \partial H_g$!!!

- Basis of $K_A^{red}(H_g)$. Let Γ be a spine of H_g (trivalent graph that H_g deformation retracts to)



For a r-admissible coloring of Γ, let Y_c be the skein obtained by





Thm: $\{ Y_c \mid c\ r\text{-adm coloring of } \Gamma \}$ form a basis of $K_A^{red}(\Sigma)$

Pf. 1) $\{Y_c\}$ span. This comes from the fact (again) that 1_n = linear comb of Jones-Wenzl idempotents.

2) $\{Y_c\}$ are linearly independent. This is a consequence of the following property that $\{Y_c\}$ are orthogonal w.r.t. a bilinear form $\langle , \rangle_{YM}$ on $K_A^{red}(\Sigma_g)$.

$\square$

Yang-Mills trace and bilinear form.

Consider $i: H_g \hookrightarrow D(H_g) \cong \#_g S^2 \times S^1$.

$$i_*: K_A^{red}(H_g) \longrightarrow K_A^{red}(S^2 \times S^1) \cong \mathbb{C}$$

$$\Rightarrow \quad YM: K_A^{red}(H_g) \longrightarrow \mathbb{C}.$$

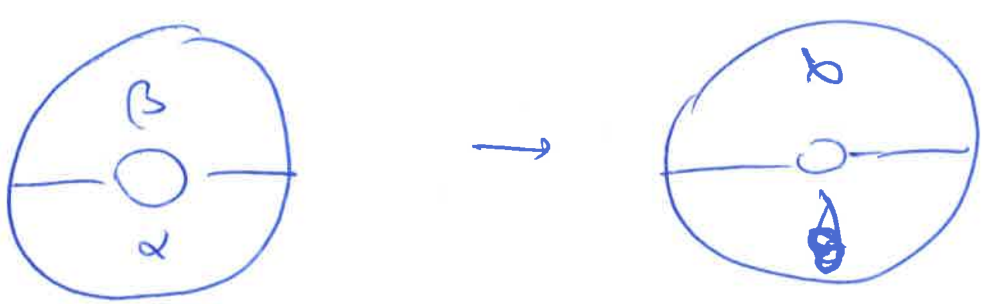
Def: $\langle , \rangle_{YM} : K_A^{red}(H_g) \times K_A^{red}(H_g) \longrightarrow \mathbb{C}$

$$(\alpha, \beta) \longmapsto YM(\alpha \cup \beta).$$

- $\langle , \rangle_{YM}$ is symmetric, because

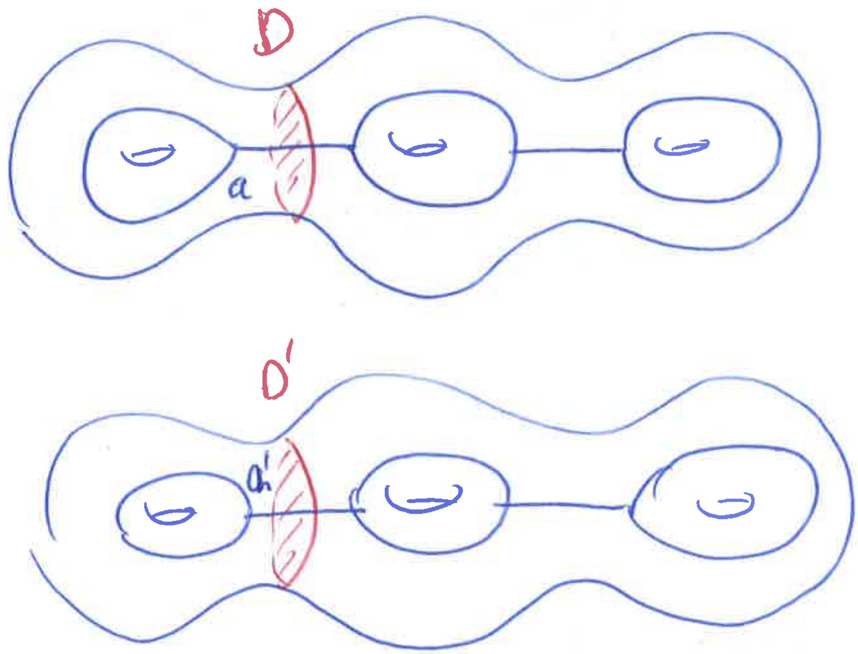
$H_g \cong \Sigma \times I$ for some punctured Σ,

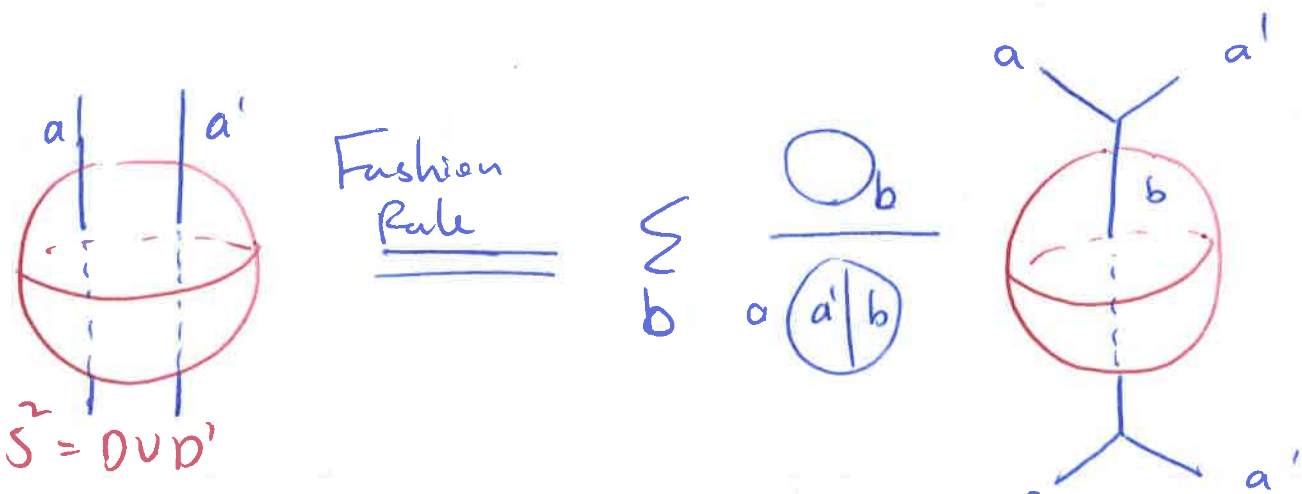
hence $D(H_g) \cong \Sigma \times S^1$.

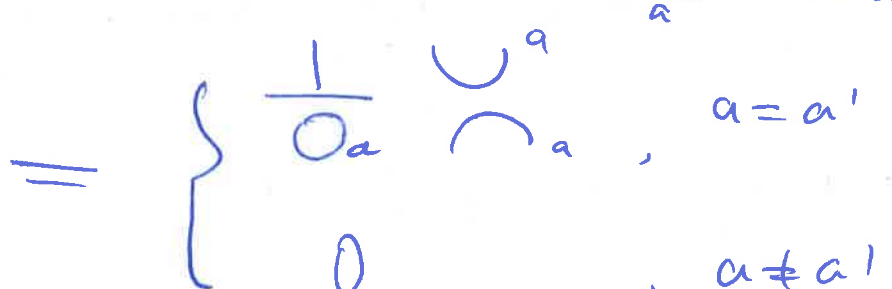


Prop: $\{Y_c\}$ are orthogonal.

Pf:

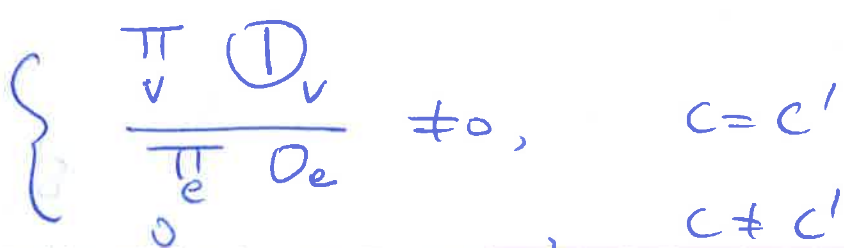






Therefore,

$\langle Y_c, Y_{c'} \rangle =$



∎

Thm (Roberts). If M_1, M_2 connected and $\partial M_1 = \partial M_2$, then $K_A^{red}(M_1) \cong K_A^{red}(M_2)$.

For the pf, need

Lemma: $\exists$ framed links $L_1 \subset M_1$, $L_2 \subset M_2$ s.t.

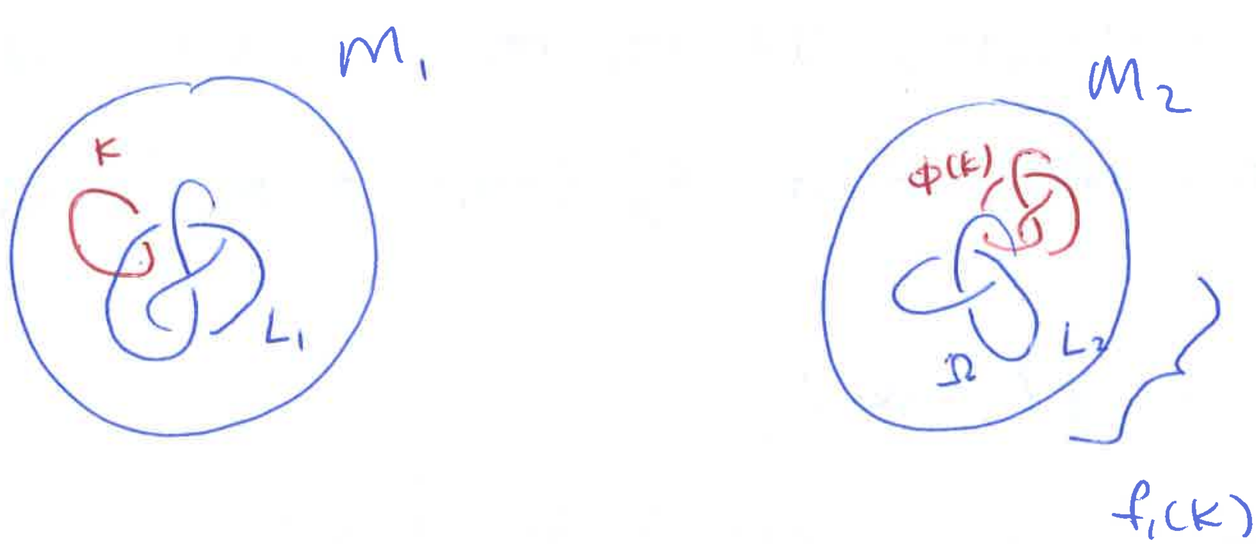
(1) $\exists$ homeomorphism $\phi: M_1 \setminus L_1 \to M_2 \setminus L_2$,

(2) $(M_1)_{L_1} = M_2$ and $(M_2)_{L_2} = M_1$.

Pf of Thm: Define $f_1: K_A^{red}(M_1) \to K_A^{red}(M_2)$ as follows. For $K \subset M_1$, isotope K so that $K \cap L_1 = \emptyset$ so $K \subset M_1 \setminus L_1$ (and $\phi(K) \subset M_2 \setminus L_2$) let

$$f_1(K) = \phi(K) \cup \Omega_{L_2},$$

where $\Omega = \mu \sum_{n=0}^{r-2} \langle e_n \rangle e_n$ and Ω_{L_2} is the cabling of each component of L_2 by Ω.



For the well-definition of f_1, we need to check $f_1(K) = f_1(K')$, where

We notice that K and K' differ by a handle-slid over γ. Therefore, $\phi(K)$ and $\phi(K')$ differ by a handle-slid over $L_2 = \phi(\gamma)$ (because $(M_2)_{L_2} = M_1$).

$\Rightarrow f_1(K) = \phi(K) \cup \Omega_{L_2} = \phi(K') \cup \Omega_{L_2} = f_1(K')$.

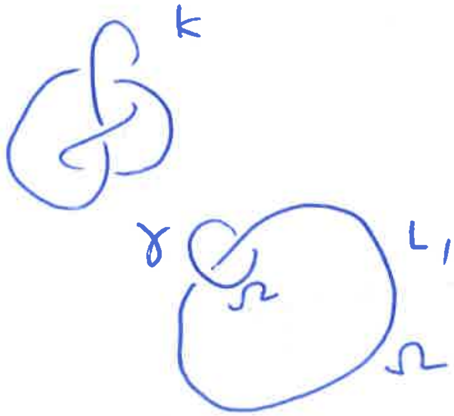
$\in K_A^{red}(M_2)$.

Let $f_2 : K_A^{red}(M_2) \longrightarrow K_A^{red}(M_1)$ be defined similarly.

Then $f_2 \circ f_1 : K_A^{red}(M_1) \longrightarrow K_A^{red}(M_2)$

$$K \longmapsto K \cup \Omega_\gamma \cup \Omega_{L_1} = K.$$

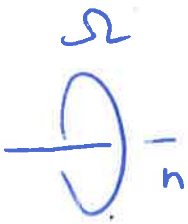
Recall



$=$

$\in K_A^{red}(M_1)$

by



$$= \begin{cases} \mu, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Indeed,

$$\Omega_\gamma \cup \Omega_{L_1} = \mu \Omega_\gamma = \mu^2 \eta = 1. \qquad \square$$

Now we consider the case that A is not a root of 1. The same argument can prove that

Thm. ① $K_A(S^2 \times S^1) \cong \mathbb{C}$

② $K_A(M \# N) \cong K_A(M) \otimes K_A(N)$

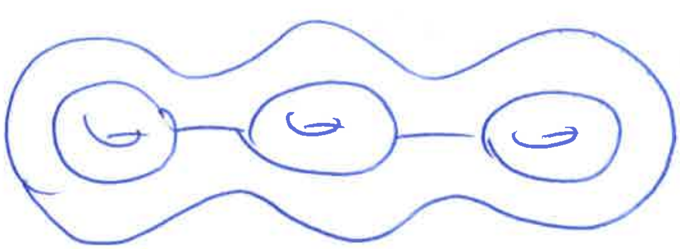
③ $K_A(\#_k S^2 \times S^1) \cong \mathbb{C}$.

The Yang-Mills inner product can be defined similarly as

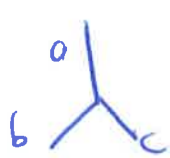
$$\langle , \rangle_{YM} : K_A^{red}(H_g) \times K_A^{red}(H_g) \longrightarrow \mathbb{C}$$

$$(\alpha, \beta) \longmapsto YM(\alpha \cup \beta).$$

. Orthogonal basis:



Γ spine, c admissible coloring of Γ, ie



$a+b \geq c$, $b+c \geq a$, $a+c \geq b$

$a+b+c$ even

Thm: $\{ Y_c \mid c \text{ adm} \}$ form a orthogonal basis of $K_A(H_g)$ w.r.t. $\langle , \rangle_{YM}$.