Lecture 13. Reshetikhin - Turaev TQFT.

Category $\mathcal{C} = \mathcal{C}ob_2^{p_1}$:

Objects: oriented closed surfaces Σ.

Morphisms: equivalence classes of cobordisms w/ structures.

Cobordism w/ structures:

(M, L, n), where $M: \Sigma_1 \to \Sigma_2$, $L \subset M$ framed link, and $n \in \mathbb{Z}$.

$(M: \Sigma_1 \to \Sigma_2$ means $\partial M = -\Sigma_1 \amalg \Sigma_2)$.

$(M_1, L_1, n_1) \sim (M_2, L_2, n_2)$ iff $\exists$ homeomorphism
ori. pres.

$\phi: M_1 \to M_2$ rel. ∂ s.t. $\phi(L_1) = L_2$, and

$n_1 = n_2$.

Notations:

Let $V : \mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}$ be a functor s.t $V(\phi) = \mathbb{C}$.

- If $(M, L, n) : \Sigma_1 \to \Sigma_2$, denote by

$$Z_{(M,L,n)} \doteq V(M, L, n) : V(\Sigma_1) \to V(\Sigma_2)$$

- If $(M, L, n) : \phi \to \Sigma$, then denote by

$$Z(M, L, n) = Z_{(M,L,n)}(1) \in V(\Sigma).$$

- If $(M, L, n) : \phi \to \phi$, ie, M is a closed cobordism, then denote by

$$\langle (M, L, n) \rangle \doteq Z(M, L, n) \in \mathbb{C}$$

Def: $V: \mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}$ is a <u>quantization functor</u> ③

if ① $V(\emptyset) = \mathbb{C}$,

② $V(-\Sigma) = \overline{V(\Sigma)}$ conjugate vector space,

$Z(-(M, L, n)) = \overline{Z(M, L, n)}$ conjugate map.

③ $\exists$ non-degenerate hermitian sesquilinear

form $\langle , \rangle_\Sigma$ on $V(\Sigma)$ s.t. if $\partial M_1 = \partial M_2 = \Sigma$, then

$$\langle Z(M_1, L_1, n_1), Z(M_2, L_2, n_2) \rangle = \langle (M_1 \cup_\Sigma -M_2, L_1 \cup L_2, n_1 + n_2) \rangle$$

hermitian: $\langle y, x \rangle = \overline{\langle x, y \rangle}$

sesquilinear: $\langle ax, by \rangle = a\bar{b} \langle x, y \rangle$.

Rm. In some literature e.g. Atiyah, require $\langle , \rangle_\Sigma$ to be positive definite.

Def: $V: \mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}$ is cobordism generated if $\forall \Sigma$, the vectors $Z(M)$, $\partial M \in \Sigma$, generate $V(\Sigma)$.

Prop: If V is cobordism generated (C.G.), then there are the following natural maps.

① (duality map) $D: V(-\Sigma) \to V(\Sigma)^*$.

② (multiplication) $\mu: V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \amalg \Sigma_2)$.

pf. ① For $Z(N, L, n) \in V(-\Sigma)$, define

$$D(Z(N,L,n))((M, L', m)) = \langle (M \cup_{\Sigma} N,\ L \cup L',\ m+n) \rangle$$

Since V is C.G, D linearly extends to $V(-\Sigma)$

② For $Z(M, L, m) \in V(\Sigma_1)$, $Z(N, L', n) \in V(\Sigma_2)$, have $Z(M \amalg N,\ L \cup L',\ m+n) \in V(\Sigma_1 \amalg \Sigma_2)$. Define

$$\mu(Z(M,L,m) \otimes Z(N,L',n)) \doteq Z(M \amalg N,\ L \cup L',\ m+n)$$

Since V is C.G, μ linearly extends to $V(\Sigma_1) \otimes V(\Sigma_2)$. □

Def: A cobordism generated quantization functor $V: \mathcal{C} \to Vect_{\mathbb{C}}$ is a TQFT if the natural maps D and μ are isomorphisms.

Trace formula for TQFT

Let $(M, L, n) : \Sigma \longrightarrow \Sigma$ and let $\# M_{\Sigma}$ be the closed bordism obtained by identifying the two copies of Σ. Then

$$\boxed{\langle (M_{\Sigma}, L, n) \rangle = tr\left(Z_{(M,L,n)} : V(\Sigma) \to V(\Sigma) \right).}$$

In particular,

$$\boxed{\langle (\Sigma_g \times S^1, \phi, 0) \rangle = \dim_{\mathbb{C}} V(\Sigma_g)}$$

Recall a quantization functor $V: \mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}$ gives an invariant $\langle (M, L, n) \rangle$ for closed M.

~~Def~~ Prop: $\langle\ \rangle$ is multiplicative that

$$\langle (M_1, L_1, n_1) \sqcup (M_2, L_2, n_2) \rangle = \langle M_1, L_1, n_1 \rangle \langle M_2, L_2, n_2 \rangle$$

and $\langle \emptyset \rangle = 1$; and is involutive that

$$\langle (-M, -L, -n) \rangle = \overline{\langle (M, L, n) \rangle}.$$

Thm (BHMV)

Given a multiplicative and involutive invariant $\langle\rangle$ of closed bordism in $\mathcal{C}$, $\exists!$ cobordism generated quantization functor on $\mathcal{C}$ extending $\langle\rangle$.

Universal construction:

Let $\mathcal{V}(\Sigma)$ be the $\mathbb{C}$-vector space freely generated by the set of cobordisms in $\mathcal{C}$

$$(M, L, n) : \phi \to \Sigma$$

Given the invariant $\langle \rangle$, define $\langle , \rangle_\Sigma$ on $\mathcal{V}(\Sigma)$ by $\langle (M, L, m), (N, L', n) \rangle_\Sigma \doteq \langle (M \cup_\Sigma (-N), L \cup -L', m+n) \rangle$, and extend sesquilinearly to a hermitian on $\mathcal{V}(\Sigma)$.

Let $R \subset \mathcal{V}(\Sigma)$ be the radical of $\langle , \rangle$, ie,

$x \in \mathcal{V}(\Sigma)$ s.t $\langle x, y \rangle = 0$ for all $y \in \mathcal{V}(\Sigma)$.

Then $\langle , \rangle_\Sigma$ descents to $V(\Sigma) \doteq \mathcal{V}(\Sigma)/R$, a non-degenerate form, still denoted by $\langle , \rangle_\Sigma$.

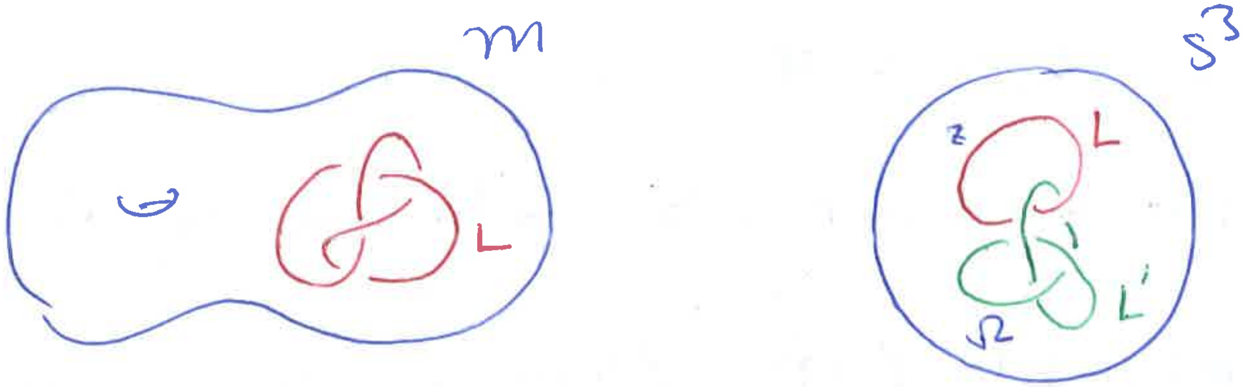
If $(M, L, n) : \Sigma_1 \to \Sigma_2$, then the assignment

$$(M', L', n') \longmapsto (M' \cup_\Sigma M, L' \cup L, n'+n)$$

defines a linear map $Z_{(M,L,n)} : V(\Sigma_1) \to V(\Sigma_2)$ s.t (V, Z) is a quantization functor.

RT-invariant for (M, L, n)

Let $L' \subset S^3$ s.t. $M = (S^3)_{L'}$ and $\sigma(L') = n$.



Def:

$$I_r(M, L) = \mu \langle \underbrace{\Omega, \dots, \Omega}_{L'}, \underbrace{z, \dots z}_{L} \rangle \cdot \langle \Omega \rangle_{u_+}^{-\sigma(L')}$$

Eg. $I_r(S^3, L) = \mu \langle L \rangle$, Kauffman bracket of L

Def: $$\boxed{\langle (M, L, n) \rangle = I_r(M, L) \cdot x^n}$$

$$= \mu \langle \underbrace{\Omega, \dots, \Omega}_{L'}, \underbrace{z, \dots, z}_{L} \rangle,$$

where $x = \langle \Omega \rangle_{u_+}$. Rm $|x| = 1$.

Thm

① $V(\Sigma_g) \cong K_A^{red}(H_g)$

② If $\Sigma = \coprod_k \Sigma^k$, $H = \#_k H^k$ s.t. $\partial H = \Sigma$,

then $V(\Sigma) \cong K_A^{red}(H)$.

Cor: V is a TQFT.

Pf of Cor: Because $K_A^{red}(H_g)$ is finite dimensional, and $\langle , \rangle_\Sigma$ is non-degenerate, $D: V(-\Sigma) \to V(\Sigma)^*$ is an iso. The multiplication $\mu: V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \sqcup \Sigma_2)$ is iso because $K_A^{red}(M \# N) \cong K_A^{red}(M) \otimes K_A^{red}(N)$.

Pf of Thm: Define $\phi: K_A^B(H_g) \to V(\Sigma_g)$ by

$$L \longmapsto (H_g, L, 0).$$

Claim: Φ is surjective!

Indeed, for $(M, L, n) \in V(\Sigma)$, let $L' \subset H_g$ be s.t. $M = (H_g)_{L'}$ and $\sigma(L') = n$.

Still denote by L the link in H_g that goes to $L \subset M$ under surgery. Then

$$\Phi(\Omega_{L'} \cup L) = (M, L, n) \in V(\Sigma).$$

For this, need to check

$$\langle (H_g, \Omega_{L'} \cup L) - (M, L, n), * \rangle = 0, \ \forall * \in V(\Sigma).$$

This follows from the following

<u>Surgery property</u>: If M, N closed, $M = N_{L'}$, then

$$\langle (M, L, \sigma(L')) \rangle = \langle (N, \Omega_{L'} \cup L, 0) \rangle.$$

Pf: True for S^3 by def. Then use that N is obtained by surgery from S^3 to reduce to S^3 case.

To see the kernel of Φ, let $H = H_g \subset S^3$ s.t. $H' = S^3 \setminus H \cong H_g$.

Define

$$\langle , \rangle_K : K_A(H) \times K_A(H') \longrightarrow \mathbb{C} \quad \text{by}$$

$$(\alpha, \beta) \longmapsto \langle \alpha \cup \beta \rangle$$

Kauffman bracket.

We check that "Φ sends $\langle , \rangle_K$ to $\langle , \rangle_\Sigma$."

Let M_1, M_2 s.t. $\partial M_1 = \partial M_2 = \Sigma$. Let $L_1' \subset H$ and $L_2' \subset H'$ s.t. $M_1 = (H)_{L_1'}$ and $-M_2 = (H')_{L_2'}$.

If $n_1 = \sigma(L_1')$ and $-n_2 = \sigma(L_2')$, then

$$\phi(\Omega_{L_1'} \cup L_1) = (M_1, L_1, n_1) \quad \text{and}$$

$$\phi(\Omega_{L_2'} \cup L_2) = (-M_2, -L_2, -n_2).$$

On the one hand, we have

$$\langle (M_1, L_1, n_1), (M_2, L_2, n_2) \rangle = \langle (M_1 \cup_\Sigma -M_2, L_1 \cup -L_2, n_1 - n_2) \rangle.$$

On the other hand,

$$\langle \Omega_{L_1'} \cup L_1, \Omega_{L_2'} \cup L_2 \rangle = \langle \Omega, \cdots, \Omega; z, \cdots, z \rangle_{L_1' \cup L_2' \cup L_1 \cup -L_2}$$

$$= \langle m_1 \cup_{\Sigma} -m_2, L_1 \cup -L_2, n_1 - n_2 \rangle,$$

because $m_1 \cup_{\Sigma} -m_2 = (S^3)_{L_1' \cup L_2'}$.

By the lemma below,

$$K_A(H_g) \Big/ \text{radical of } \langle , \rangle_K \cong V(\Sigma).$$

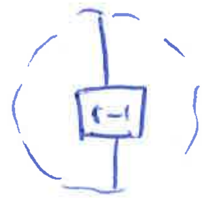
Lemma: If $\phi : V_1 \to V_2$ is surjective s.t

$\langle u, v \rangle_{V_1} = \langle \phi(u), \phi(v) \rangle_{V_2}$ and $\langle , \rangle_{V_2}$ is

non-degenerate, then

$$V_1 \Big/ \text{radical of } \langle , \rangle_{V_1} \cong V_2.$$

Recall $K_A^{red}(H_g) = K_A(H_g) \big/ ③④$,

where ③:  $= 0$

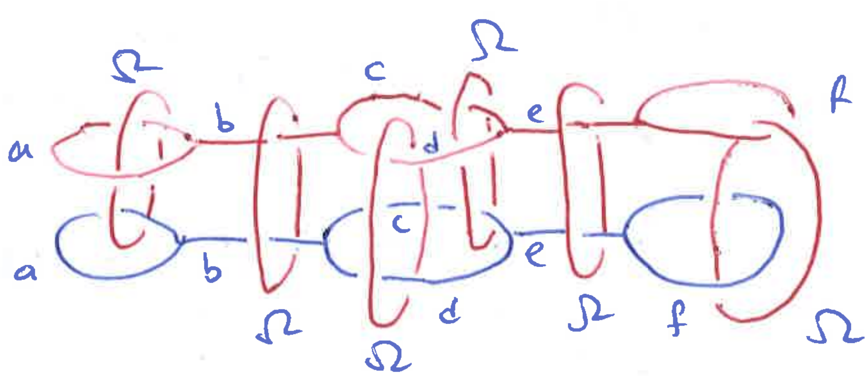
④:  $= 0$, (a, b, c) non r-adm.

Let R_K = radical of $\langle , \rangle_K$ and let

$$R_{red} = \operatorname{span}_{\mathbb{C}} \{ ③, ④ \}.$$

Goal: $R_K = R_{red}$.

- $R_{red} \subset R_K$ because ③, ④ vanish in $K_A(S^3)$
- $Y_c \notin R_K$ because $\langle Y_c, L_c \rangle \neq 0$, where

$L_c =$ 

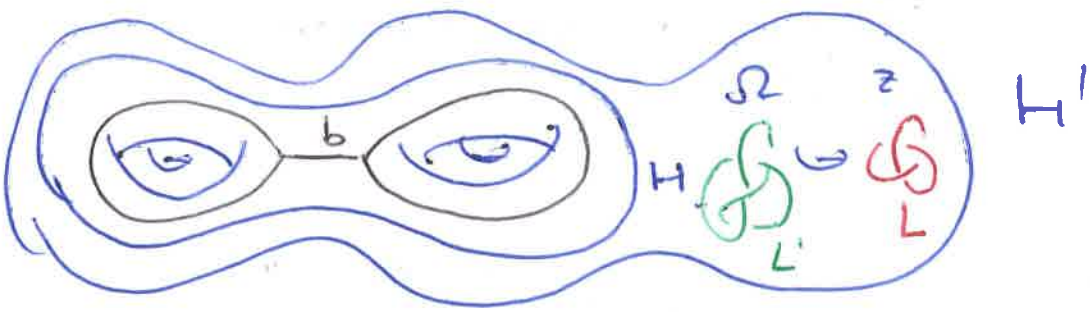
Fushion Rule.

$$= \frac{\prod_{\Omega} \theta}{\prod_{e} \bigcirc}$$

What is $Z_{(m,L,n)} : K_A^{red}(H) \longrightarrow K_A^{red}(H')$

for $(m, L, n) : \Sigma \to \Sigma'$?

Choose any embedding $i : H \hookrightarrow H'$.



Let $m' = H' \setminus H$, then $\partial m' = \partial m$.

$\exists\, L' \subset m'$ s.t. $m = (m')_{L'}$ and $\sigma(L') = n$.

Still denote by L the link in m' that goes to L under surgery.

Define $$Z_{(m,L,n)} : K_A^{red}(H) \longrightarrow K_A^{red}(H')$$
$$b \longmapsto b \cup \Omega_{L'} \cup L.$$

What is $\langle , \rangle_\Sigma$ on $K_A^{red}(H)$?

Recall Yang - Mills $\langle , \rangle_{YM} : K_A^{red}(H) \times K_A^{red}(H) \to \mathbb{C}$.

$\searrow K_A^{red}(D(H)) \cong$

Answer: $\langle \alpha . \beta \rangle_\Sigma = \langle \alpha . \bar{\beta} \rangle_{YM}$.

$$\langle \alpha . \beta \rangle_\Sigma = \langle (H, \alpha), (H, \beta) \rangle_\Sigma$$

$$= \langle H \cup (-H), \alpha \cup \bar{\beta} \rangle$$

$$= \langle D(H), \alpha \cup \bar{\beta} \rangle$$

Recall. If $M \cong (S^3)_{L'}$, then

$$K_A^{red}(M) \cong K_A^{red}(S^3) \cong \mathbb{C}$$

$$L \longmapsto L \cup \Omega_{L'} \mapsto \langle (M, L) \rangle$$

$$= \langle \alpha . \bar{\beta} \rangle_{YM}.$$