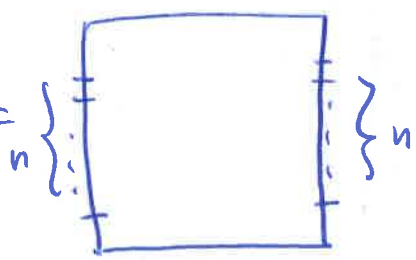
Lecture 6: Temperly – Lieb algebras and the Jones – Wenzl idempotents.

For $r \geq 3$, $r \in \mathbb{N}$, let

$$\boxed{A_r = e^{\frac{\pi i}{2r}}}$$

Rm. Can also consider any primitive $4r$-th root of 1,

e.g. $A = e^{\frac{k\pi i}{2r}}$, $(k, 4r) = 1$.

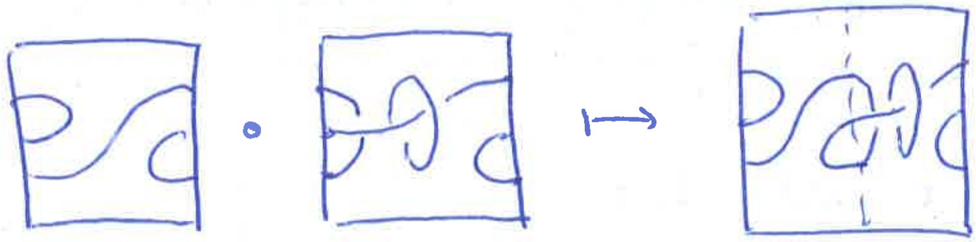
- Let $D_{n,n} =$  be a square w/ $2n$ points on a pair of opposite edges, n for each.

Def: The n-th Temperly – Lieb algebra TL_n is the $\mathbb{Z}[A_r^{\pm 1}]$-module generated by (isotopy classes of) link diagrams in $D_{n,n}$ modulo relations ① ②, where

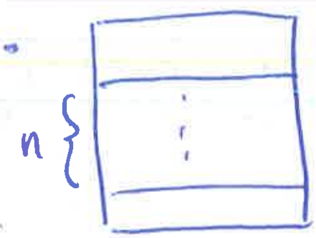
① Kauffman bracket skein rel'n: 

② Framing rel'n: $\bigcirc \sqcup D = (-A^2 - A^{-2})\, D$

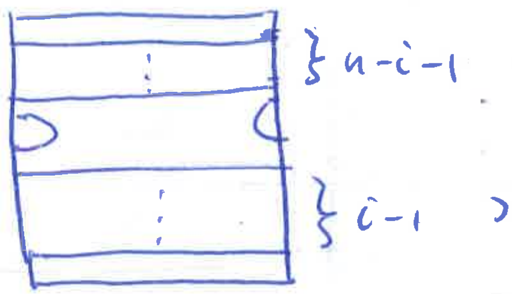
- Product "$\circ$" on TL_n: juxtaposing



Rm. $(TL_n, \circ)$ is the skein algebra of $D_{n,n}$.

- 

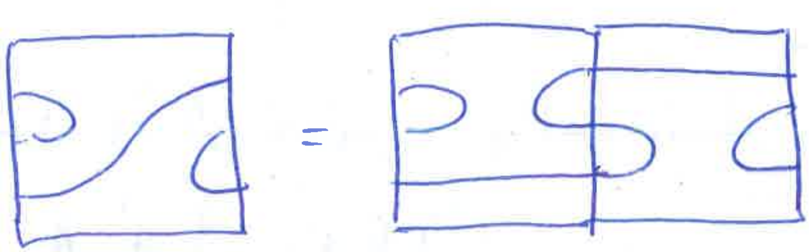
is the unit of $\circ$, hence is denoted by 1 (or 1_n).

- Let b_i = 

, $1 \le i \le n-1$.

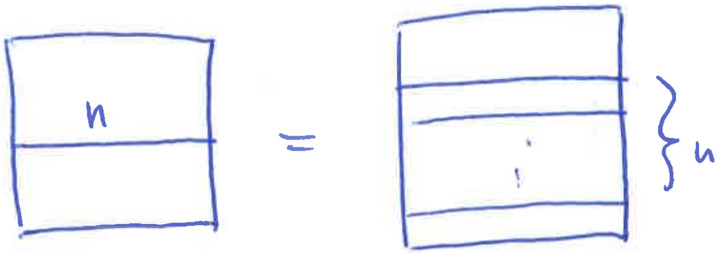
Prop: $(TL_n, \circ)$ is generated by $\{1, b_1, \dots, b_{n-1}\}$

Pf: By ①②, get embedded diagram w/out ○, which is a product of 1 and b_i's.

Eg:  $= b_1 \cdot b_2$

□

Notation:

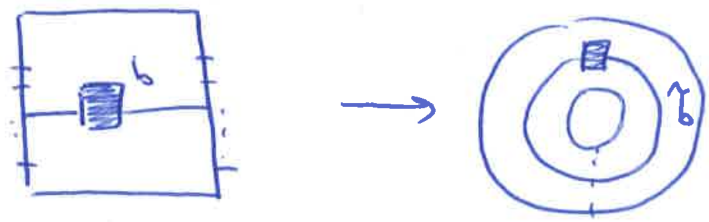


n parallel line segments.

Jones-Wenzl idempotents !!!

Idea: element $f_n \in TL_n$ that "closes up" to $e_n \in \mathcal{B}$.

$\sim$: $TL_n \longrightarrow \mathcal{B}$



Def/Lemma: $\forall n \in \mathbb{N}$, $\exists!$ $f_n \in TL_n$, called the n-th Jones-Wenzl idempotent, s.t.

(i) $f_n b_i = b_i f_n = 0$, $1 \le i \le n-1$ (f_n kills "turn arounds")

(ii) $f_n - 1$ belongs to subalgebra B generated by $\{b_1, \dots, b_{n-1}\}$ ($f_n = 1 +$ product of b_i's)

(iii) $\tilde{f}_n = e_n$ (recall $e_0 = 1$, $e_1 = z$, $e_{n+1} = z e_n - e_{n-1}$ Chebyshev poly of 2nd type.)

Immediate consequence:

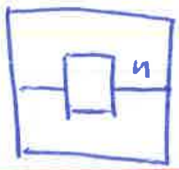
$$f_n \cdot f_n = f_n$$

that's why it's called "idempotent".

pf: $(f_n - \mathbb{1})\, f_n \overset{(ii)}{=} (\text{product of } b_i\text{'s}) \cdot f_n \overset{(i)}{=} 0$.

pf of Uniqueness.

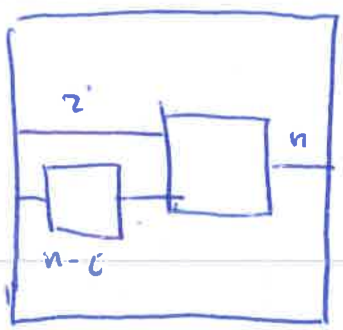
By (i)(ii), $1 - f_n$ is the unit of $\mathcal{B}$, hence is unique.

Notation: $f_n =$ 

For existence, we need.

Lemma: If f_k exists for $k \le n$, then

(1) !!



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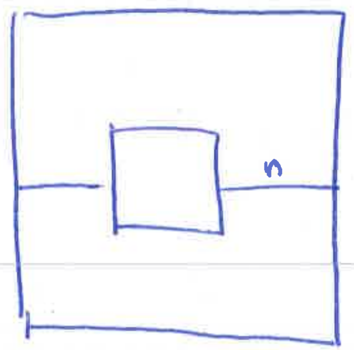
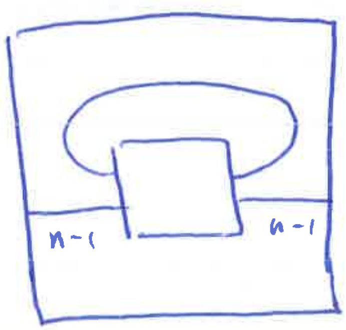


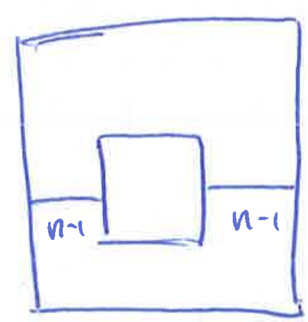
, $0 \le i \le n$

important formulas

(2)

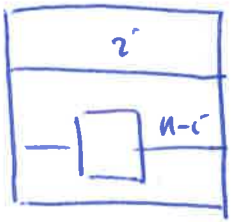


$$= \frac{\langle e_n \rangle}{\langle e_{n-1} \rangle}$$



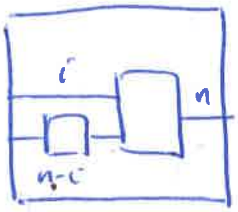
f_n "eats" every thing!!!

Pf of (1). By (i)



$- q \in B$,

and by (ii)



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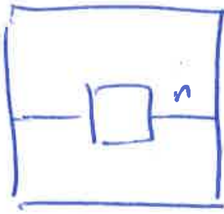
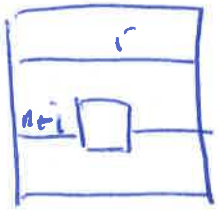
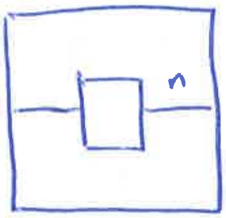


$= \Big($



$- q \Big) \cdot f_n = 0$.

(2).



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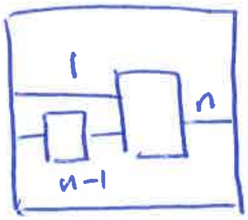
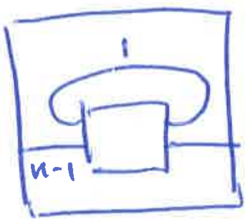


$\Rightarrow$



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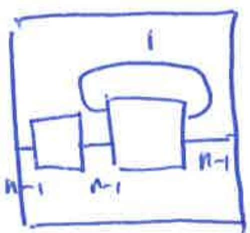
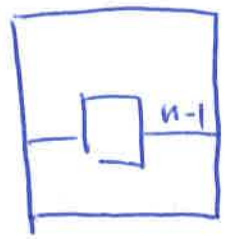
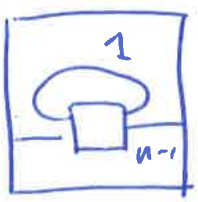


$\overset{(i)}{=} m \cdot$



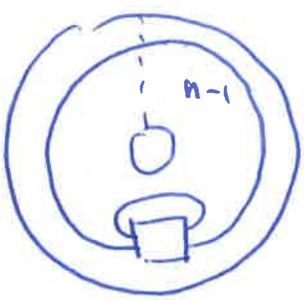
, where

m is s.t



$= m \cdot q +$ product of b_is.

Close up $\Rightarrow$



$= m$

Take $\langle \ \rangle$ $\Rightarrow$ $\langle e_n \rangle = m \langle e_{n-1} \rangle$, hence

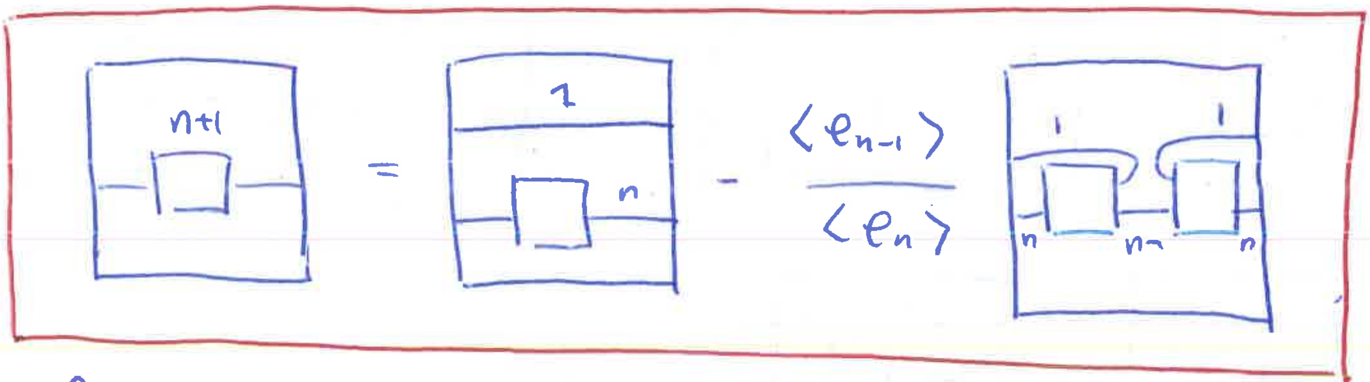
$$m = \frac{\langle e_n \rangle}{\langle e_{n-1} \rangle}.$$

$\square$

Existence of JWI: by induction.

Let $f_1 = 1 =$  . Suppose $f_1, \dots, f_n$ exist.

Define f_{n+1} by

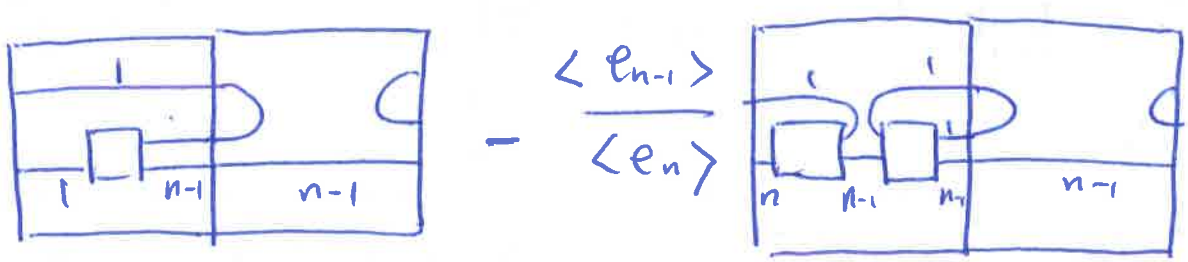


!!

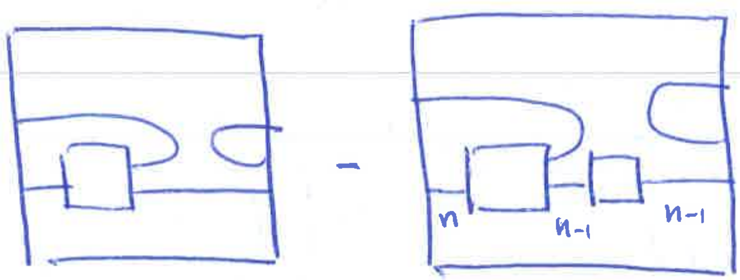
(1) follows easily from definition except

$$f_{n+1} b_n = b_n f_{n+1} = 0.$$

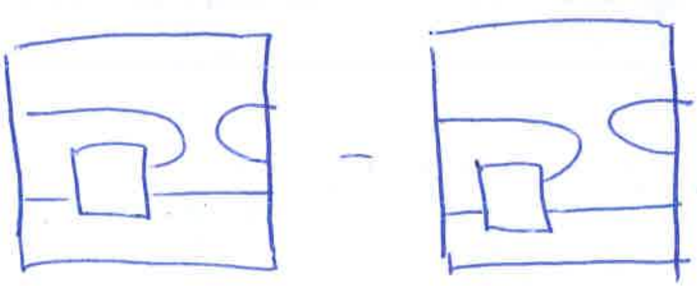
$f_{n+1} b_n =$



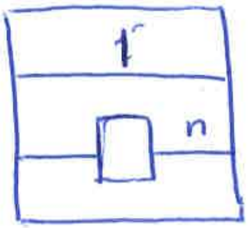
by (2)



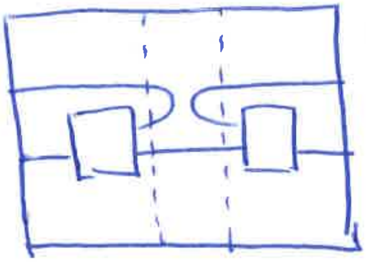
by (1)



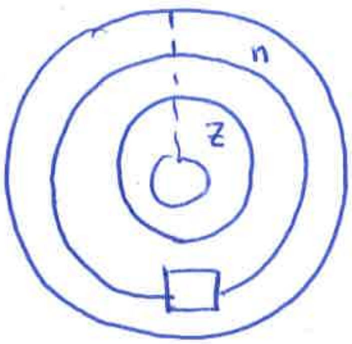
$= 0.$

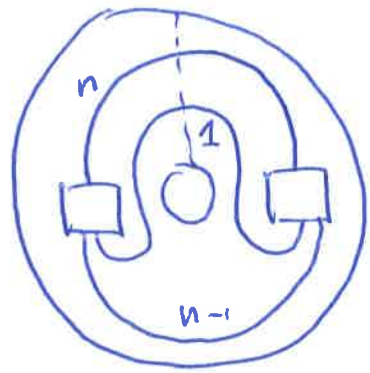
(ii) 

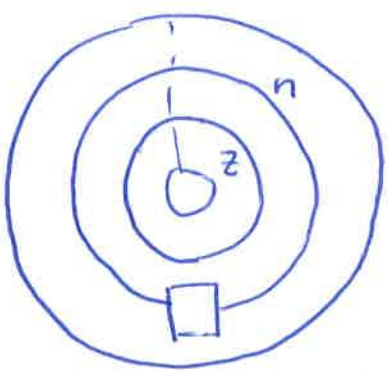
$= 1 +$ product of b_i's.

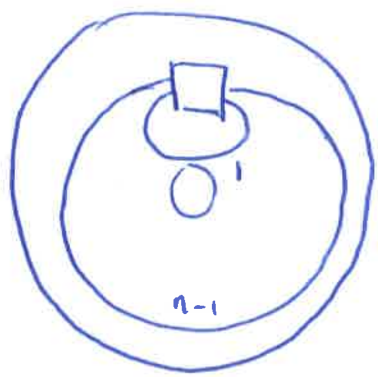

$\in B$.

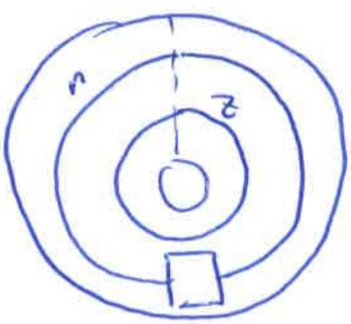
$\uparrow$
b_n

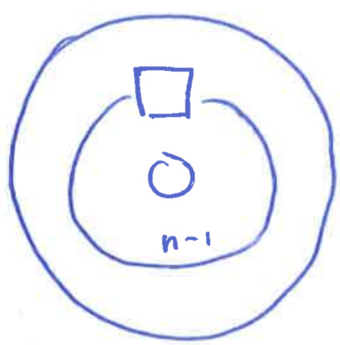
(iii) $\tilde{f}_n =$ 

$- \dfrac{\langle e_{n-1} \rangle}{\langle e_n \rangle}$ 

$\underset{\text{by (1)}}{=}$ 

$- \dfrac{\langle e_{n-1} \rangle}{\langle e_n \rangle}$ 

$\underset{\text{by (2)}}{=}$ 

$-$ 

$= z \cdot e_n - e_{n-1} = e_{n+1}$ $\square$