Quantum invariants (skein theory) Lecture 1

History

Jones mid 80's. Jones polynomial

using rep'n theory of operator alg.

(Kauffman, skein theory)

Witten. 5 years later. Witten's invariant.

reinterpretes Jones polynomial using QFT.

Reshetikhin – Turaev. (90, 91).

– colored Jones polynomial links.

– Reshetikhin - Turaev invariants 3-mfd

using rep'n theory of quantum groups.

(Lickorish, Blanchet-Hambegger-Masbaum, Vogel
↕
Kauffman, Wenzel. skein ~~theory~~ skein theory)

Turaev – Viro 92. Turaev-Viro invariant

(Kauffman – Lins. skein theory rep of quantum group).

Theorem (Turaev, Walker, Roberts)

$$TV = |RT|^2$$

skein theory.

define using triangulation; define using surgery diagram

Plan:

Why Quantum?:
1. invariants.
2. rep'ns of MCG.
3. attack classical problems.

Week 1 Jones polynomial, Colored Jones. today

2 Reshetikhin – Turaev.

3 Turaev - Viro.

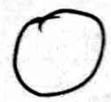
4 Theorem of T-W-R.

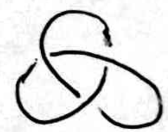
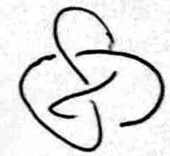
or something else?

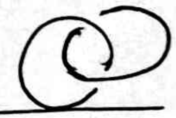
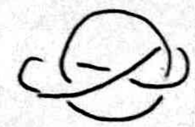
Def: A <u>link</u> in S^3 is a smooth embedding

$$L: \coprod_k S^1 \longrightarrow S^3$$

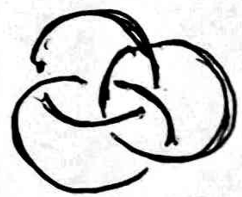
of k disjoint S^1 into S^3. If $k=1$, it is a <u>knot</u>, and is usually denoted by K.

Eg: trivial knot (unknot) .

trefoil:  figure-8. 

Hopf link:  Whitehead link: 

why Hopf?

Borromean ring: 

Two links L_1 and L_2 are <u>equivalent</u> if they are isotopic, i.e., $\exists$ smooth map $L: (\coprod_k S^1) \times [0,1] \longrightarrow S^3$ s.t.

$$L\big|_{(\coprod_k S^1)\times\{0\}} = L_1, \quad L\big|_{(\coprod_k S^1)\times\{1\}} = L_2$$

and $\forall t \in (0,1)$, $L\big|_{\coprod S^1 \times \{t\}}$ is smooth embedding

All knots/links above are not equivalent

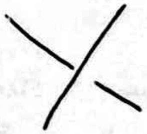
WHY??? not easy.

One way to answer it is to use <u>invariant</u>

To define invariants, we start w/

<u>link diagrams</u>

Def. A <u>link diagram</u> is a graph in $\mathbb{R}^2$ w/ over and under crossings.



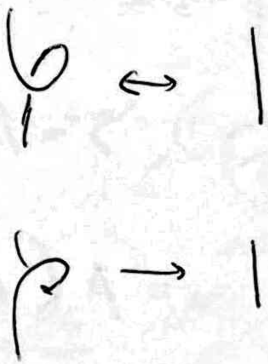
(projection of a link into $\mathbb{R}^2$).

Q. when do two diagrams represent the same link?

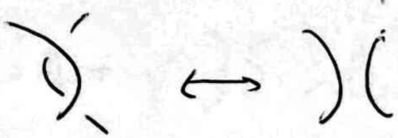
Thm (Reidemeister)

Two diagrams represent the same link if and only if one could be obtained from the other by a sequence of the following three moves.

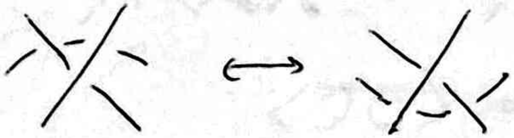
Reidemeister Move I.



RM II.



RM III.



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idea of pf:

projection of the isotopy $L_1 \overset{L}{\sim} L_2$ may have the following singularities.

all others are generic pts

Kauffman bracket.

Given a diagram D of L, $\langle L \rangle$ is a Laurent polynomial in A, and B defined by the following two rules.

~~① Kauffman bracket skein rel'n~~

~~$\langle \times \rangle = A$~~

① Kauffman bracket skein relation.

$$\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$$

possitive resolution negative resolution

go along lower one, turn right.

② Framing relation

$$\langle \bigcirc \rangle = -A^2 - A^{-2}.$$

Eg. $\langle \text{Hopf link} \rangle = A \langle \cdot \rangle + A^{-1} \langle \cdot \rangle$

$= A\left(A\langle \bigcirc\bigcirc \rangle + A^{-1}\langle \bigcirc \rangle\right)$

$+ A^{-1}\left(A\langle \bigcirc \rangle + A^{-1}\langle \bigcirc\!\!\bigcirc \rangle\right)$

$= A\left(A(-A^2-A^{-2})^2 + A^{-1}(-A^2-A^{-2})\right)$

$+ A^{-1}\left(A(-A^2-A^{-2}) + A^{-1}(-A^2-A^{-2})^2\right) = A^6 + A^2 + A^{-2} + A^{-6}$

Q. Is $\langle D \rangle$ invariant under RM I, II, III ?

RM II. $\langle \cdot \rangle = A \langle \cdot \rangle + A^{-1} \langle \cdot \rangle$

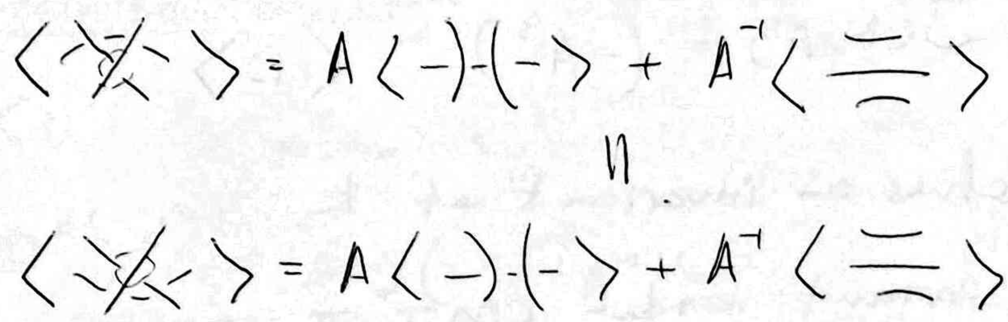
$= A\left(A\langle \cdot \rangle + A^{-1}\langle \cdot \rangle\right)$

$+ A^{-1}\left(A\langle)(\rangle + A^{-1}\langle \cdot \rangle\right)$

$= (A^2 + A^{-2})\langle \asymp \rangle + \overset{-A^2-A^{-2}}{\langle \bigcirc\asymp \rangle} + \langle)(\rangle$

$= \langle)(\rangle$

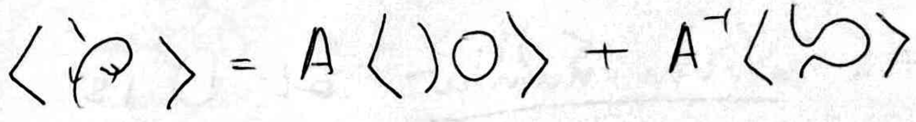
RM III.



$$\langle \text{---} \rangle = A\langle \text{---} \rangle + A^{-1}\langle \text{---} \rangle$$

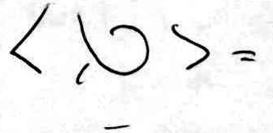
$$\langle \text{---} \rangle = A\langle \text{---} \rangle + A^{-1}\langle \text{---} \rangle$$

RM I.



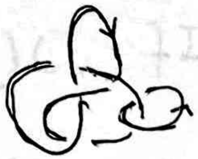
$$= \left(A(-A^2 - A^{-2}) + A^{-1}\right)\langle \,|\, \rangle$$

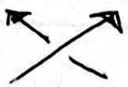
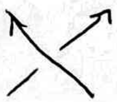
$$= -A^3 \langle \,|\, \rangle. \quad (\neq \langle \,|\, \rangle).$$



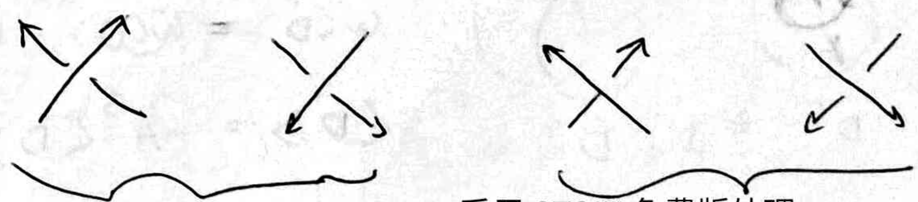
$$= -A^{-3}\langle \,|\, \rangle$$

Let D be a diagram of L, and give D arbitrarily an orientation



A crossing is possitive  if as you go along the lower, the upper goes from left to right, and is negative  if otherwise.

Rm. If D is a diagram of a knot K, then being possitive/negative is independent of the orientation.



possitive

negative

Def. Let D be a diagram of a knot K. Then the <u>writhe number</u> of D is

$$w(D) = \#\{\text{possitive crossings}\} - \#\{\text{negative crossings}\}.$$

Def\Thm (Jones). The following

$$J(K,A) \doteq (-A^3)^{-w(D)} \langle D \rangle$$

defines an invariant of K, i.e., is invariant under RM I, II, III.

pf. If D' and D differ by a RM II, or III then $w(D') = w(D)$ and $\langle D' \rangle = \langle D \rangle$.

If D' and D differ by RM I, then

D' $\qquad$ D.

$w(D') = w(D) - 1$ &

$\langle D' \rangle = -A^{-3} \langle D \rangle$.

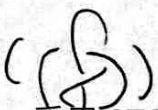
$$(-A^3)^{-w(D')} \langle D' \rangle = (-A^3)^{-w(D)+1} (-A^{-3}) \cdot \langle D \rangle = (-A^3)^{-w(D)} \langle D \rangle.$$

Conj (Jones).

If $J(K,A) = J(O,A) = -A^2 - A^2$,

then $K = O$.

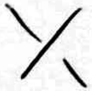
Ex. Calculate.

$J($$)$, $J($$)$.

last time Jones polynomial <u>lecture 2</u>

<u>Kauffman bracket</u>: D diagram of L.

$\langle D \rangle$ is defined by

①  $= A\ )(\ + A^{-1} \asymp$

② $O = -A^2 - A^{-2}$.

<u>Jones polynomial</u>: D diagram of K.

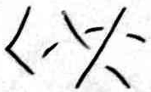
$$J(K, A) := (-A^3)^{-w(D)} \langle D \rangle.$$

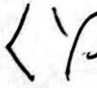
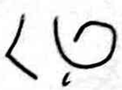
is an invariant of K, i.e., invariant under RM I, II, III.

<u>Normalized Jones polynomial</u>

$$J'(K, A) = \frac{J(K, A)}{J(O, A)} = \frac{J(K, A)}{-A^2 - A^{-2}}$$

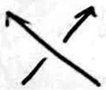
$\langle D \rangle$ is invariant under RM II and III.

$\langle$  $\rangle = \langle\ )(\ \rangle$. $\langle$  $\rangle = \langle$  $\rangle$,

and $\langle$  $\rangle = -A^3 \langle\ |\ \rangle$. $\langle$  $\rangle = -A^{-3} \langle\ |\ \rangle$.

<u>Writhe number</u>: D diagram of <u>knot</u> K.

$w(D) = \#\{+\text{ crossings}\} - \#\{-\text{ crossings}\}$

 "+"  "−".

Eg.: $J\langle O, A \rangle = -A^2 - A^{-2}$.

HW ($J\langle$  $, A \rangle = A^{18} - A^{10} - A^{6} - A^{2}$

$J\langle$  $, A \rangle = -A^{10} - A^{-10}$.

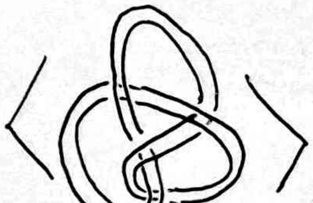
$J'(O) = 1$

$J'($  $) = -A^{16} + A^{12} + A^{4}$

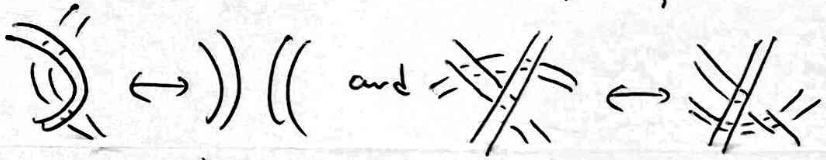
$J'($  $) = A^{8}$ [illegible] A^{-4} [illegible] A^{-8}

Colored Jones polynomial:

Naive idea: consider



· Invariant under RM II, III, since



are composition of RM II, III only

$-A^4\langle \;\; \rangle = (A^8+1)\langle || \rangle$.

$\langle \;\; \rangle = A\langle \;\; \rangle + A^{-1}\langle \;\; \rangle$

$\langle \;\; \rangle = -A^3\langle \;\; \rangle = -A^3(-A^2-A^{-2})$.

$\langle \;\; \rangle = A\langle \;\; \rangle + A^{-1}\langle || \rangle = A(-A^2-A^{-2}) + A^{-1}\langle || \rangle$

How about RM I ?

$\langle \;\; \rangle = A\langle \;\; \rangle + A^{-1}\langle \;\; \rangle$

$= -A^4\langle \;\; \rangle + A^2\langle \;\; \rangle$

$= A^8 \cancel{(A^8+1)} \langle || \rangle - A^8 + 1$.

"not a eigenvector of $\;\;$ "

idea: consider

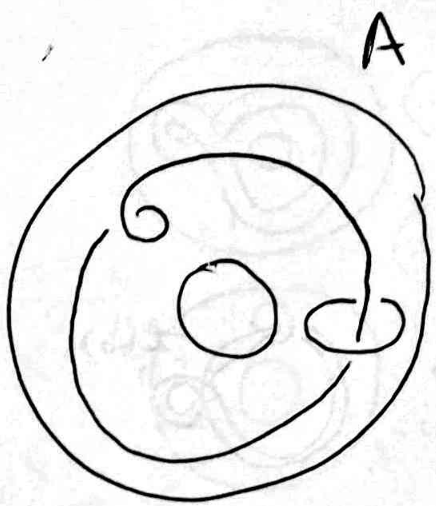
then

$\langle \;\; -1 \rangle = A^8 \langle || - 1 \rangle$

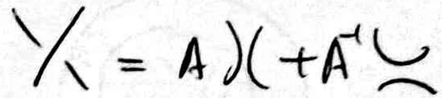
Let R be the ring of Laurent poly. in A, i.e.

$$R = \mathbb{Z}[A^{\pm}].$$

Let $\mathcal{B}$ be the R-~~algebra~~ module generated by link diagrams in an annulus $\mathbb{A}$, modulo the relations ①, ②.

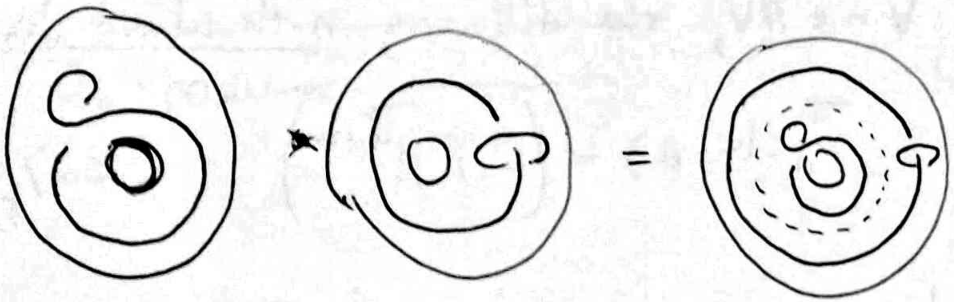


①

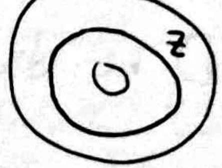


② $\bigcirc = -A^2 - A^{-2}$.

Algebra structure on $\mathcal{B}$ is induced by



- $\emptyset$ is consider as $\in \mathcal{B}$, and is the unit of multiplication, hence is denoted by 1.

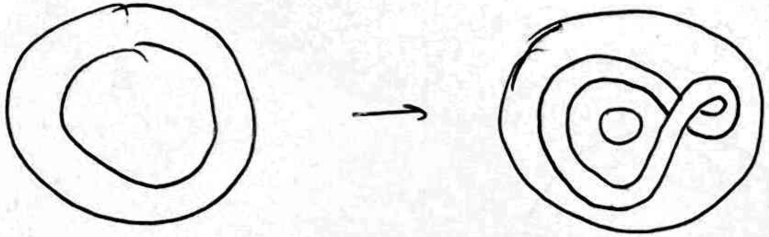
Thm. Let $z \in \mathcal{B}$ be .

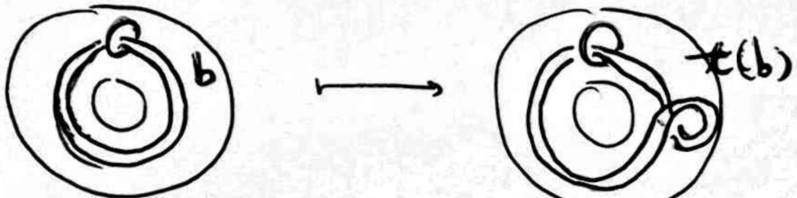
Then

$$\mathcal{B} \cong \mathbb{Z}[A^{\pm}][z].$$

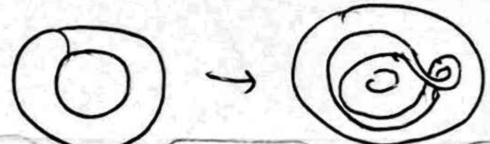
Rk. $\mathcal{B}$ is the Kauffman bracket skein algebra of $\mathbb{A}$.

<u>Twist operator</u> $t: \mathcal{B} \to \mathcal{B}$ is (the linear operator) induced by





• t^{-1} is induced by

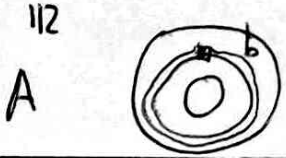


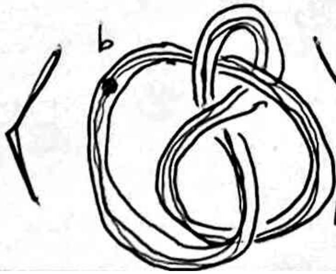
Let D be diagram of K, and let $b \in \mathcal{B}$.

The <u>cabling of D by b</u>, denoted by $\langle b \rangle_D$, is (the Kauffman bracket of the diagram obtained by) putting b on D w/ the black board framing.

$K: S^1 \times [0,1] \longrightarrow \mathbb{R}^2$

$\quad\ \ \shortparallel$





$= \langle b \rangle_D$

<u>Key lemma</u>: Let e_n be the element of $\mathcal{B}$ defined inductively by

$e_0 = 1, \ e_1 = z$ and $e_n = z e_{n-1} - e_{n-2}$

Then

$$t(e_n) = (-1)^n A^{n^2+2n} e_n.$$

Def/Thm (Reshetikhin-Turaev, Wenzel)

$\forall n \in \mathbb{N}$, the following <u>n-th colored Jones poly.</u>

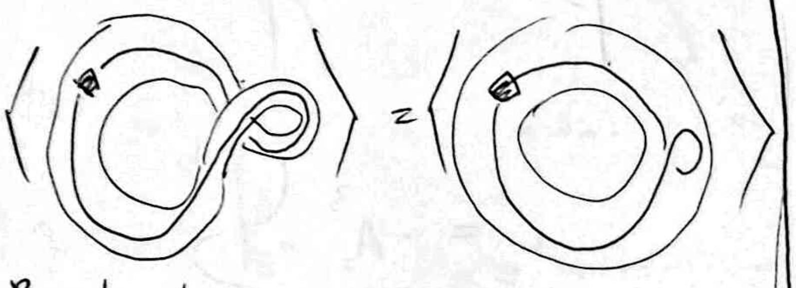
$$J_n(K, A) = \left((-1)^n A^{n^2+2n}\right)^{-w(D)} \langle e_n \rangle_D$$

defines an invariant of K, w/

$$J_1(K, A) = J(K, A)$$

Pf. Let D' be from D by ρ. Then

① $\langle e_n \rangle_{D'} = \langle t(e_n) \rangle_D$



By Key Lemma:

$= (-1)^n A^{n^2+2n} \langle e_n \rangle_D$

② $w(D') = w(D)+1$

$$J_n(O,A) = \langle e_n \rangle_O$$

$$= \langle z\, e_{n-1} - e_{n-2} \rangle_O$$

$$= (-A^2 - A^{-2}) \langle e_{n-1} \rangle_O - \langle e_{n-2} \rangle_O$$

$\langle e_0 \rangle_O = 1$.

$\langle e_1 \rangle_O = -A^2 - A^{-2}$.

By induction: $\langle e_n \rangle_O = (-1)^n \dfrac{A^{2n+1} - A^{-2n+1}}{A^2 - A^{-2}}$

Normalized colored Jones

$$J'_n(K,A) = \frac{J_n(K,A)}{J_n(O,A)}$$

Rm. Let $q = A^2$, then

quantum integer

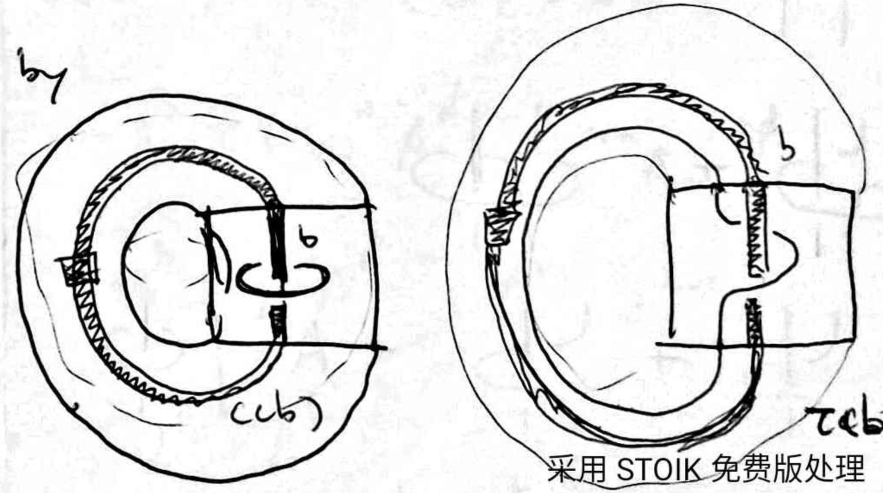
$$\langle e_n \rangle = (-1)^n [n+1]$$

$$= (-1)^n \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}}$$

Pf of Key Lemma:

Consider two operators $c, \tau : \mathcal{B} \to \mathcal{B}$

by



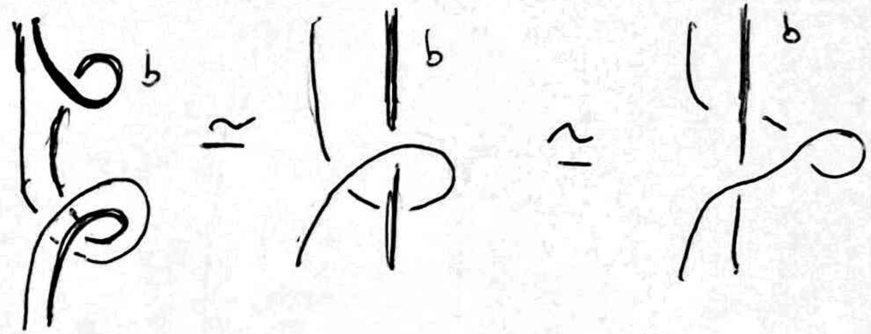
Lemma 1: $\forall\, b \in \mathcal{B}$

(i) $\tau z t^{-1}(b) = -A^{3}\tau(b)$

(ii) $c(zb) = A^{-2} z\, c(b) + (1-A^{4})\tau(b)$

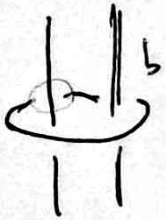
(iii) $\tau(zb) = A^{2} z\, \tau(b) + (1-A^{-4})\, c(b)$

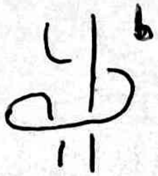
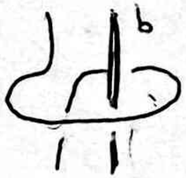
Pf (i):



$= -A^{3}$ 

(ii)



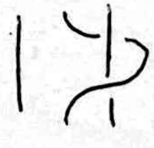
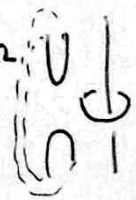
$= A$  $+ A^{-1}$ 

$= -A^{4}$  $+$  $+ A^{-2}$ 

$= A^{2} z.c(b) + (1-A^{4})\tau(b)$.

(iii)

$= A$  $+ A^{-1}$ 

$= A^{2}$  $+$  $+ A$  $+ A^{-2}$ 

$= A^{2} z\, \tau(b) + (1-A^{-4})\, c(b)$.

Lemma 2: Let $\lambda_n = -A^{2n+2} - A^{-2n-2}$, $\mu_n = (-1)^n A^{n^2+2n}$.

Then

(i) $\tau(z^n) = A^{2n} z^{n+1} + \cdots$

(ii) $c(z^n) = \lambda_n z^n + \cdots$

(iii) $t(z^n) = \mu_n z^n + \cdots$

pf. (i)

$$\tau(z^{n+1}) = A^2 z\,\tau(z^n) + (1-A^{-4})\,c(z^n)$$

$$\overset{\text{induction}}{=} A^{2n+2} \cdot z^{n+1} + \cdots$$

(ii) $c(z^{n+1}) = A^{-2} z\, c(z^n) + (1-A^4)\,\tau(z^n)$

$$= (1-A^4)\cdot A^{2n} z^{n+1} + \lambda_n A^{-2} z^{n+1} + \cdots$$

$$= \lambda_{n+1} z^{n+1} + \cdots$$

(iii) Let $u = t(z^n)$. Then

$$t(z^{n+1}) = t z t^{-1}(u) = -A^3 \tau(u)$$

$$\overset{\text{induction}}{=} -A^3 \tau\left((-1)^n A^{n^2+2n} z^n + \cdots\right)$$

$$\overset{\text{induction}}{=} -A^3 (-1)^n A^{n^2+2n} \cdot A^{2n} \cdot z^{n+1} + \cdots$$

$$= (-1)^{n+1} A^{(n+1)^2+2(n+1)} z^{n+1}$$

Lemma 3:

(i) $c(e_n) = \lambda_n e_n$

(ii) $t(e_n) = \mu_n e_n$. (key lemma) $\Longleftarrow$

• easy to see.

$$\boxed{\lambda_{n+k} + \lambda_{n-k} = -\lambda_{k-1}\lambda_n, \ \forall k}$$

• $c(z^n b) = -\lambda_0 z c(zb) + (2+\lambda_1 - z^2) c(b)$

$= A^{-2} z c(zb) + (1-A^4)\tau(zb)$

$= A^{-2} z c(zb) + (1-A^4)\left(A^2 z \tau(b) + (1-A^{-4}) c(b)\right)$

$= -\lambda_0 z c(zb) - A^2 z c(zb) + (2+\lambda_1) c(b) + (1-A^4) A^2 z \tau(b)$

Lemma 1(ii)

$= -\lambda_0 z c(zb) + (2+\lambda_1) c(b) - A^2 \cdot z \cdot A^{-2} \cdot z \cdot c(b)$

$= -\lambda_0 z c(zb) + \cancel{(2+\lambda_1) c(b)}\ (2+\lambda_1 - z^2) c(b)$

$c(e_{n+1}) = c(z e_n - e_{n-1})$

$= c(z(z e_{n-1} - e_{n-2}) - e_{n-1})$

$= -\lambda_0 z c(z e_{n-1}) + (2+\lambda_1 - z^2) c(e_{n-1})$
$\quad - c(z e_{n-2}) - c(e_{n-1})$

$= -\lambda_0 z c(z e_{n-1}) + (\cancel{2}+\lambda_1 - z^2) c(e_{n-1})$
$\quad - c(\cancel{e_{n+1}}) - c(e_{n-3}) - c(\cancel{e_{n-1}})$

$= -\lambda_0 z c(e_n) - \lambda_0 z c(e_{n-2})$
$\quad + (\lambda_1 - z^2) c(e_{n-1}) - c(e_{n-3})$

$= -\lambda_0 \lambda_n z e_n - \lambda_0 \lambda_{n-2} z e_{n-2} + \lambda_1 \lambda_{n-1} e_{n-1} - \lambda_{n-1} z^2 e_{n-1} - \lambda_{n-3} e_{n-3}$

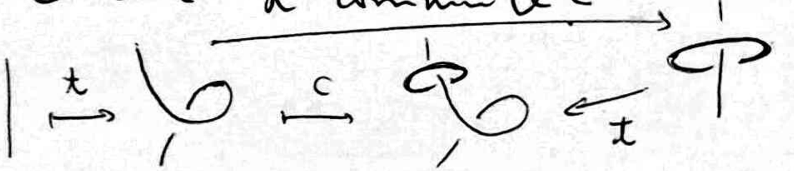
$= \lambda_{n+1} z e_n + \cancel{\lambda_{n-1} z e_n} - \lambda_0 \lambda_{n-2} z e_{n-2} - \lambda_{n+1} e_{n-1} - \cancel{\lambda_{n-3} e_{n-1}} - \cancel{\lambda_{n-1} z^2 e_{n-1}} - \cancel{\lambda_{n-3} e_{n-3}}$

$\|$

$\cancel{\lambda_{n-1} z e_{n-2}} + \cancel{\lambda_{n-3} z e_{n-2}}$

e_n is also an eigenvector of t.

• c and t commute



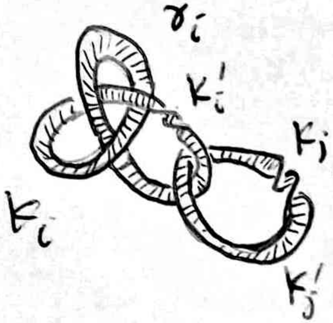
By Lemma 2 (iii).

$t(e_n$

Reshetikhin - Turaev invariants Lecture 3

- framed link.

$$L : \coprod_k \left(S^1 \times [0,1] \right) \longrightarrow S^3$$



$K_i = \gamma_i(S^1 \times \{0\})$

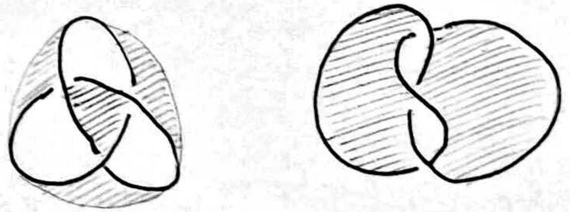
$K_i' = \gamma_i(S^1 \times \{1\})$

K_i' is a parallel copy of K_i

- Each component γ_i has a framing n_i.

- A Seifert surface of a knot $K \subset S^3$ is a connected, oriented, compact surface $S \subset S^3$ w/ $\partial S = K$.

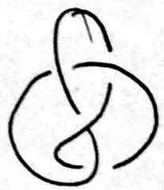
Eg.



$\cong \Sigma_{1,1}$

HW.

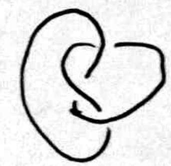
HW 2. Find a Seifert surface of

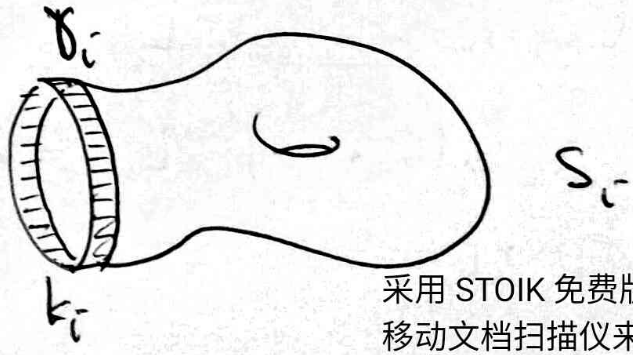


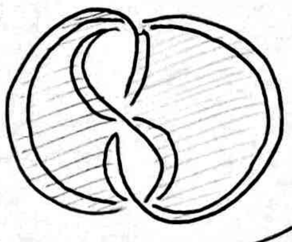
.

- γ_i is of 0-framing if it is a collar nbhd of k_i in a Seifert surface S_i of k_i.

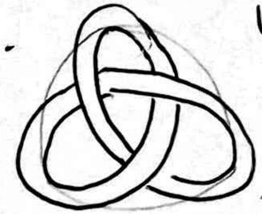
Eg.





Eg. 

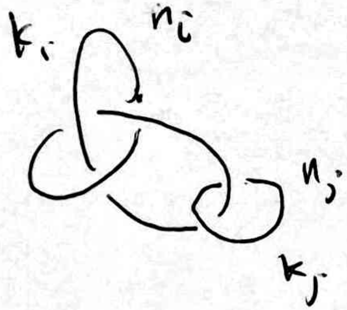
$\Longleftarrow$ last page.

Eg.  $n = -3$

- γ_i is of n-framing the algebraic intersection

$$i(K_i', S_i) = n.$$

- A framed link can also be represented by a link w/ an integer n_i on each component K_i. (Kirby diagram)



- The linking number of K_i and K_j is

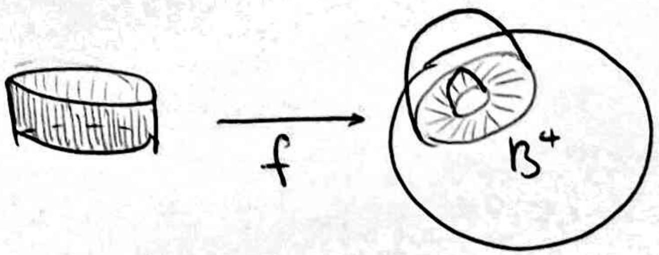
$$lk_{ij} \doteq i(K_i, S_j) = i(K_j, S_i)$$

- The linking matrix of L is $Lk = (Lk_{ij})$ defined by $L_{ii} = n_i$ and $Lk_{ij} = lk_{ij}$.

- L determines a 4-mfd X_L by attaching 2-handles to B^4 along circles K_i in L according to their framings, and hence a 3-mfd $M_L \doteq \partial X_L$.

2-handle $= D^2 \times D^2$.

$\partial(D^2 \times D^2) = (\underline{\underline{S^1 \times D^2}}) \cup (D^2 \times S^1)$.



"attaching a 2-handle".

$f: S^1 \times D^2 \longrightarrow S^3 = \partial B^4$, attaching map.

- f is determined by $f|_{S^1 \times [0,1]} : S^1 \times [0,1] \to S^3$ up to isotopy.

Rm. M_L is obtained from S^3 by doing n_i - Dehn surgery along K_i, for each i.

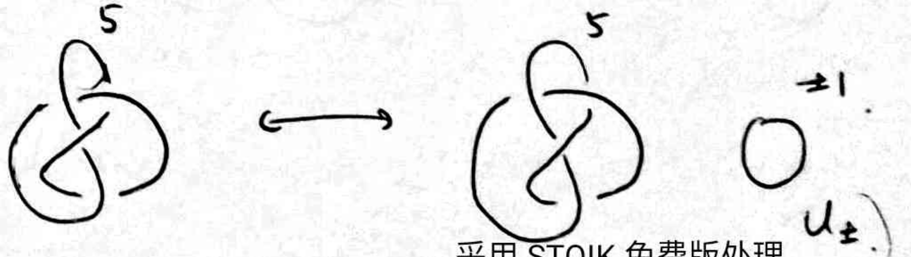
Thm (Lickorish, Wallace).

Any $M \cong M_L$ for some framed link L.

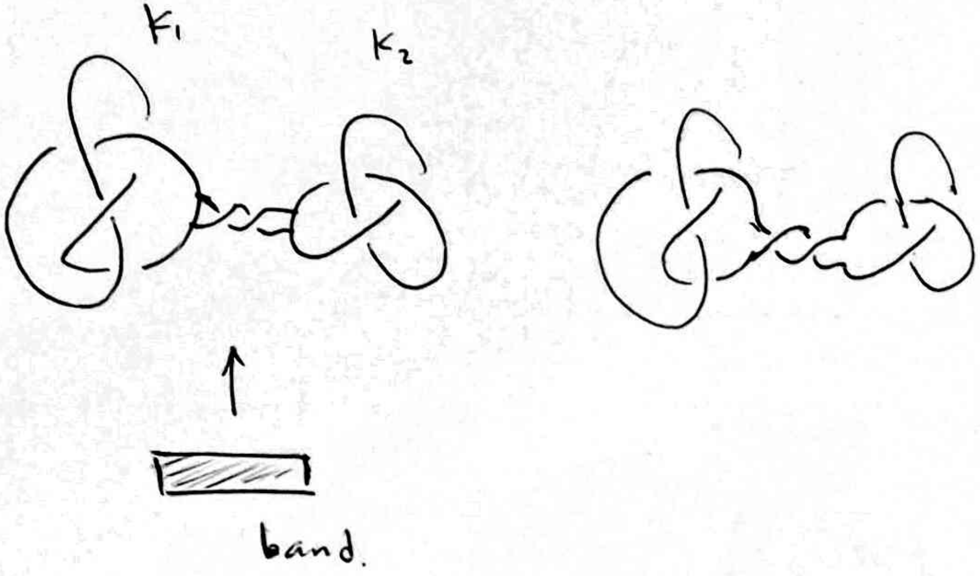
Thm (Kirby).

$M_{L_1} \cong M_{L_2}$ iff L_1 can be obtained from L_2 by a sequence of Kirby Moves KI and KII.

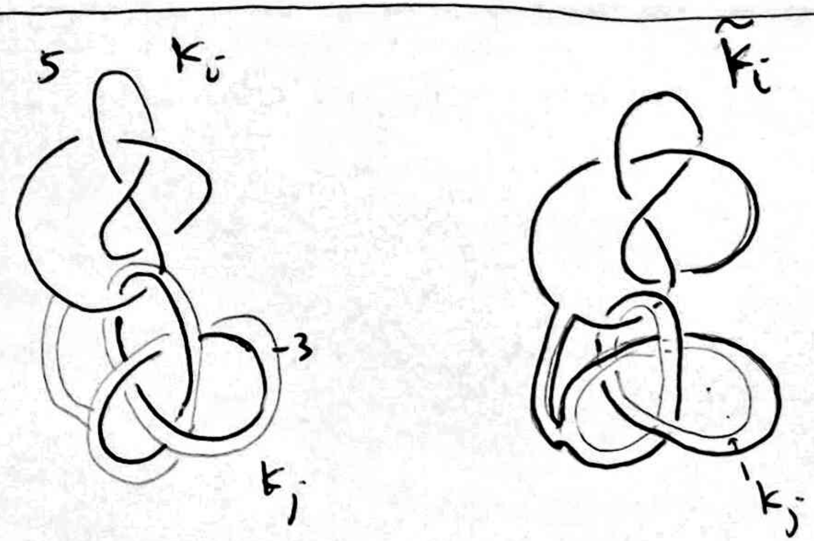
KI (blow up or blow down) changes L by adding or subtracting from L a disjoint unknotted circle w/ ± 1-framing



• Band Sum



kII (handle slid) changes L by replacing one component k_i by ~~the~~ a band sum of k_i w/ a parallel copy k_j' of another component k_j of L.

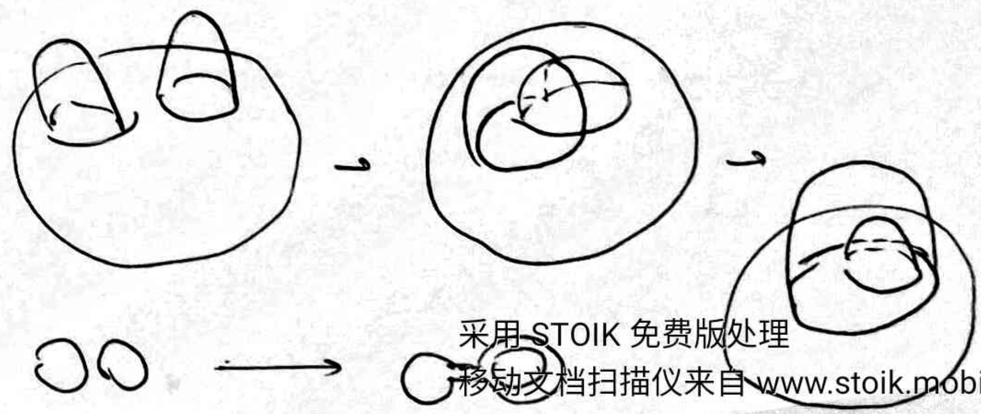


Q. what is the framing $\tilde{n}_i$ of $\tilde{k}_i$?

A. $\tilde{n}_i = n_i \pm lk_{ij} + n_j$

• ~~blow~~ blow up $\sim$ $\# \mathbb{C}P^2$ or $\# \overline{\mathbb{C}P^2}$.

• handle slid.



Lecture 4

Prop:

(i) $RT_r(M \# N) = RT_r(M) \cdot RT_r(N)$

(ii) $RT_r(-M) = \overline{RT_r(M)}$

(iii) $RT_r(S^3) = 1$.

Pf: (i) If $M = M_L$, $N = M_{L'}$, then

$$M \# N = M_{L \sqcup L'}.$$

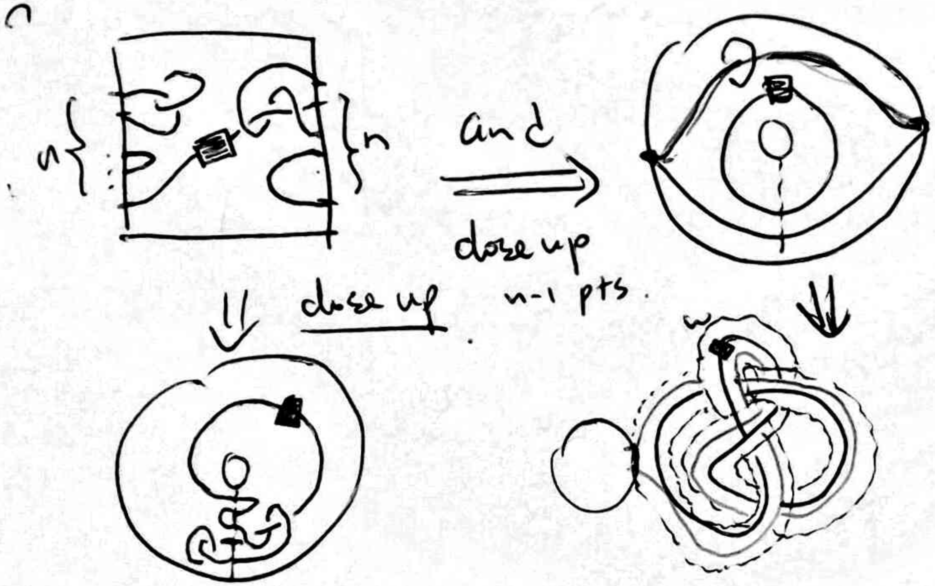
(ii) If $M = M_L$, then $-M = M_{\bar{L}}$,

where $\bar{L}$ is the mirror image of L.

(iii) $S^3 = M_{\emptyset}$.

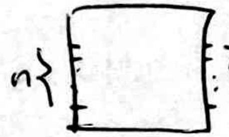
Pf of Main Thm: Consider diagrams

in

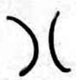


Ref: Let TL_n be the $\mathbb{Z}[A^{\pm 1}]$-module

generated by link diagrams in

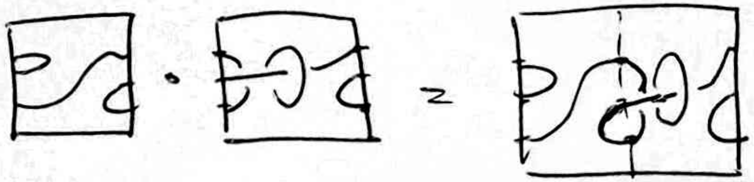


modulo relations

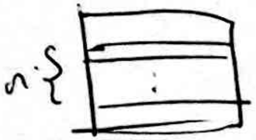
①  $= A$  $+ A^{-1}$ 

②  $= -A^2 - A^{-2}$

• Algebra structure

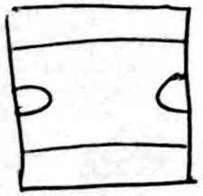


RM: $(TL_n, \circ)$ is the n-th Temperley-Lieb algebra.



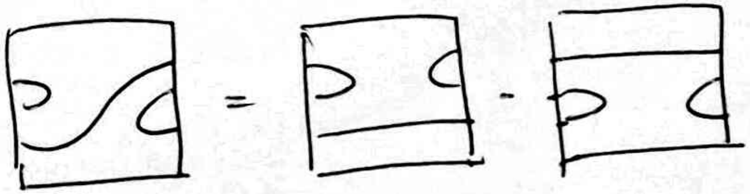
is the unit of $\circ$, hence is denoted by 1.

Prop: Let $b_i =$



. Then $(TL_n, \circ)$ is generated by $\{1, b_1, \dots, b_{n-1}\}$

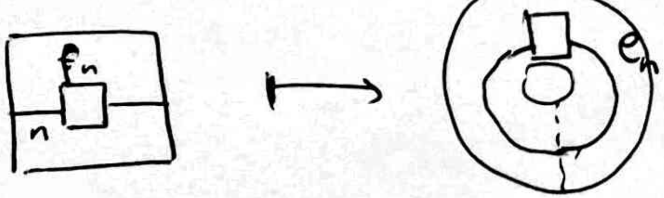
Eg:



Jones-Wenzl projector: !!!

"Intuitive def"

The element $f_n \in TL_n$ that "closes up" to $e_n \in \mathcal{B}$. $\sim : TL_n \to \mathcal{B}$.



Def/Lemma. Suppose A primitive $4r$-th root of 1. Then exists unique $f_n \in TL_n$ called the n-th Jones-Wenzl projector s.t. (i) $f_n b_i = 0 = b_i f_n$, $1 \le i \le n-1$

(ii) $f_n - 1$ belongs to algebra generated by $\{b_1, \dots, b_{n-1}\}$.

(iii) $\tilde{f}_n = e_n$.

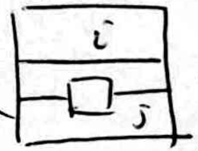
Important consequence of (i)(ii).

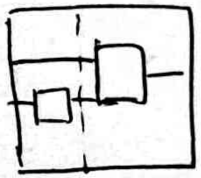
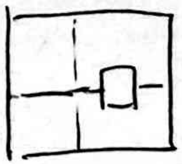
$$f_n \cdot f_n = f_n.$$

$\left((f_n - 1) f_n = 0 \Rightarrow f_n f_n - f_n = 0. \right)$

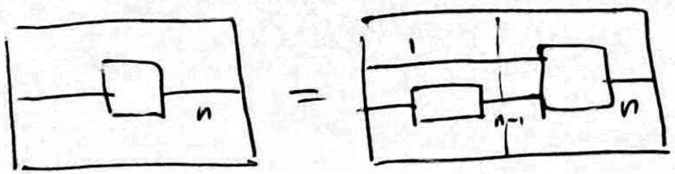
Pf. Uniqueness.

By (i)(ii), $1 - f_n$ is the unit of the alg. generated by $\{b_0, \dots, b_{n-1}\}$, hence is unique.

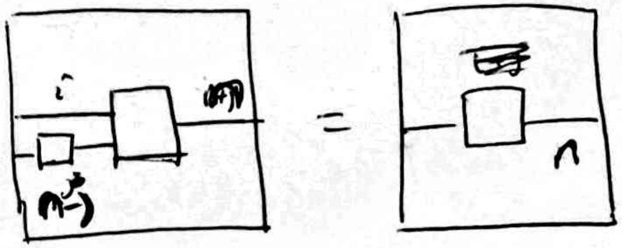
Pf: (1) By (i), $\Big($  $- 1 \Big) \in$ alg. generated by $b_0, \dots, b_{j-1}$.

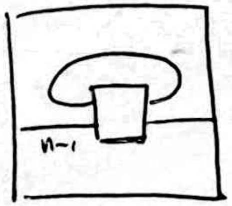
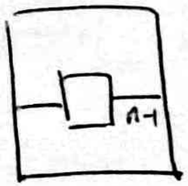
By (ii)  $-$  $= 0$.

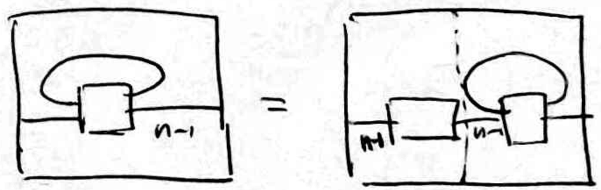
(2). By (1).

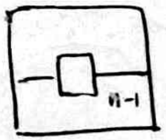


Lemma. If f_k exists for $k \le n$, then

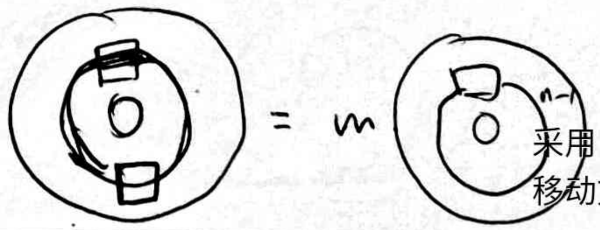
(1) 贪吃蛇  $0 \le i \le n$

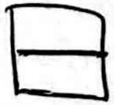
(2).  $= \dfrac{\langle e_n \rangle}{\langle e_{n-1} \rangle}$ 

$\Rightarrow$ 

By (1). RHS = multiple of ,

Close up both sides,

 $\Rightarrow m = \dfrac{\langle e_n \rangle}{\langle e_{n-1} \rangle}$

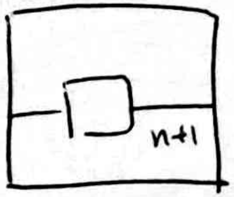
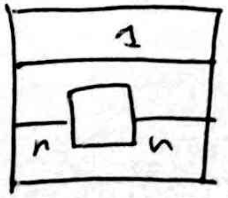
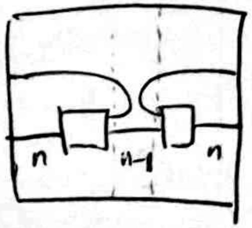
~~Induction~~ Induction. ~~Step~~ Let $f_1 =$ .

Suppose $f_1, \dots, f_n$ are defined by (i)(ii)(iii).

Define f_{n+1} by

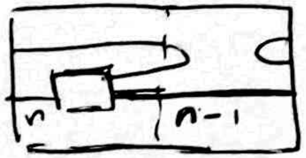
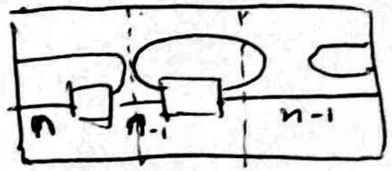
alg generated by $\{b_0, \dots, b_n\}$

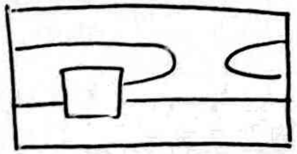
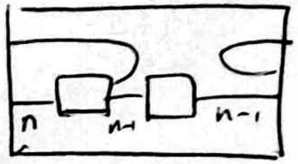
b_n

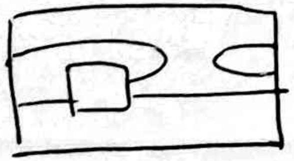
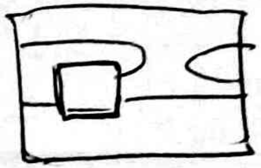
 $=$  $- \dfrac{\langle e_{n-1}\rangle}{\langle e_n\rangle}$ 

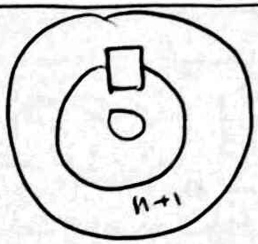
(i)(ii) follows easily, except $f_{n+1} b_n = 0$

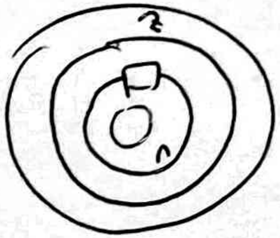
$f_{n+1} b_n$

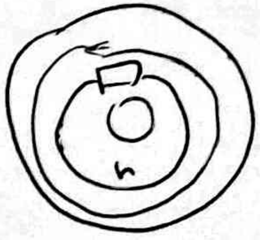
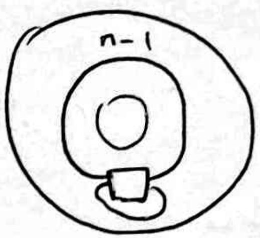
$=$  $- \dfrac{\langle e_{n-1}\rangle}{\langle e_n\rangle}$ 

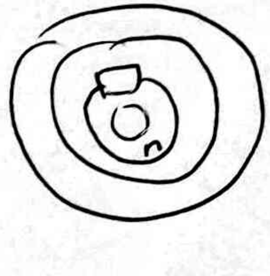
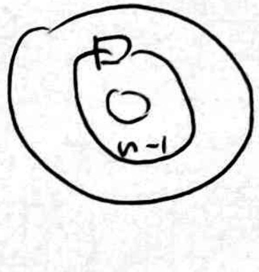
$\overset{(2)}{=}$  $-$ 

$\overset{(1)}{=}$  $-$  $= 0$.

(iii). $\tilde{f}_{n+1} =$ 

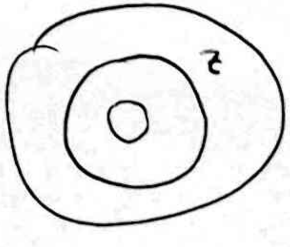
$=$  $- \dfrac{\langle e_{n-1}\rangle}{\langle e_n\rangle}$ 

$=$  $- \dfrac{\langle e_{n-1}\rangle}{\langle e_n\rangle}$ 

$=$  $-$ 

$= z e_n - e_{n-1} = e_{n+1}$. □

last time:



$\mathcal{B} \cong \mathbb{Z}[A^{\pm 1}][z]$

Kauffman bracket skein algebra of $\mathcal{A}$.

$\boxed{e_{n+1} = z e_n - e_{n-1}}$, $e_0 = 1$. $e_1 = z$.

Chebyschev polynomial.

Key lemma:

$$t^{\pm 1}(e_n) = (-1)^n A^{\pm(n^2+2n)} e_n.$$

t is twist operator induced by

- D diagram of K, $b \in \mathcal{B}$. The cabling of D by b is the Kauffman bracket of "putting b on D", denoted by $\langle b \rangle_D$.

Thm (Reshetikhin–Turaev, Wenzel).

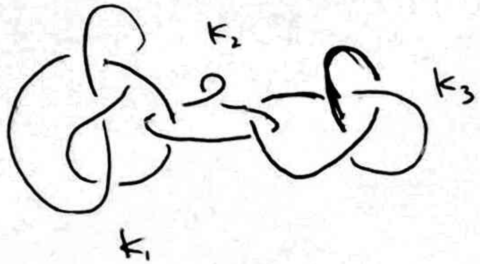
Let D diagram of K. Then the n-th Colored Jones polynomial

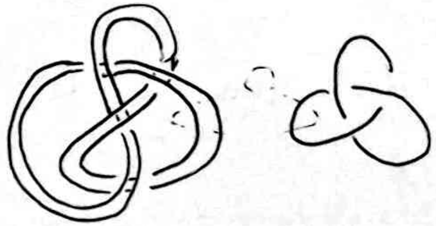
$$J_n(K, A) \doteq \left((-1)^n A^{n^2+2n}\right)^{-w(D)} \langle e_n \rangle_D$$

defines a knot invariant.

- If D is a diagram of a link L w/ n (ordered) component, and $b_1, \dots, b_n \in \mathcal{B}$. Then the cabling of D by $b_1, \dots, b_n$, denoted by $\langle b_1, \dots, b_n \rangle_D$, is the Kauffman bracket of the diagram obtained by putting b_i on the i-th component of D.

Eg:

$D =$ 

$\langle z^2, 1, z\rangle_D =$ 

- For $r \geq 3$, $r \in \mathbb{N}$, let A be a primitive $4r$-th root of 1, i.e.,

$$A = e^{\frac{2k\pi i}{4r}}, \quad (k, 4r) = 1.$$

- For simplicity, ~~let's~~ can consider $k=1$, i.e.,

$$\boxed{A = e^{\frac{\pi i}{2r}}}$$

~~Then~~ $B_r \doteq \mathbb{Z}[A^{\pm 1}][z] \subset \mathbb{C}[z]$.

Let $\omega_r \in B_r$ be defined by

$$\boxed{\omega_r = \sum_{n=0}^{r-2} \langle e_n \rangle e_n}.$$

Recall

$$\langle e_n \rangle = (-1)^n \frac{A^{2n+2} - A^{-2n-2}}{A^2 - A^{-2}}$$

Thm: (Reshetikhin–Turaev, Lickorish)

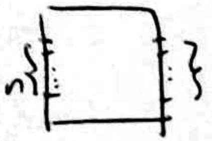
Let $M = M_L$, D be the standard diagram of L, and $b_\pm$ be the number of positive/negative eigenvalues of the linking matrix of L. Then $\forall r \geq 3$,

$$RT_r(M) \doteq \langle \omega_r, \dots, \omega_r \rangle_D \langle \omega_r \rangle_{U_+}^{-b_+} \langle \omega_r \rangle_{U_-}^{-b_-}$$

defines a complex valued invariant of M, i.e., is invariant under Kirby moves I & II.

last time: Temperley-Lieb algebra $(TL_n, \circ)$

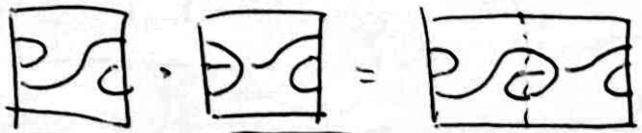
- $\mathbb{Z}[A^{\pm}]$-module generated by link diagrams

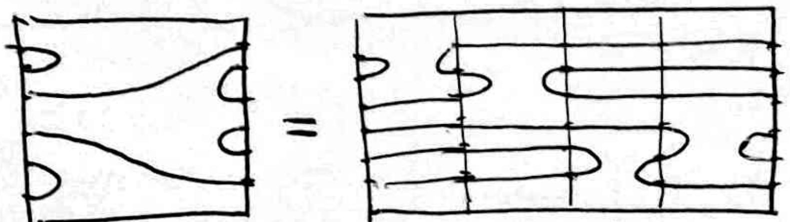
in  module

① $\times = A\,)(+ A^{-1} \asymp$

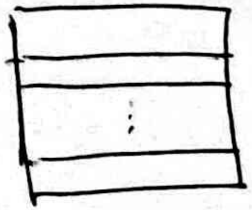
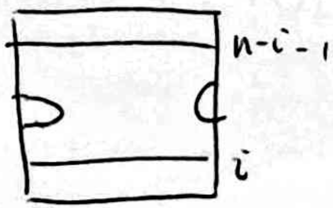
② $\bigcirc = -A^2 - A^{-2}$

- multiplication $\circ$:





$b_4 \quad b_3 \quad b_0 \quad b_1$

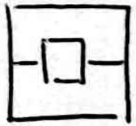
1  b_i 

Lecture 5

Prop. 1) As $\mathbb{Z}[A^{\pm}]$-module, TL_n is spanned by diagrams without crossings and disjoint circles.

2) $(TL_n, \circ)$ is generated by $\{1, b_0, \dots, b_{n-1}\}$

Jones-Wenzl projector (!!!).

$\exists!\ f_n \in TL_n$, denoted by  n, s.t

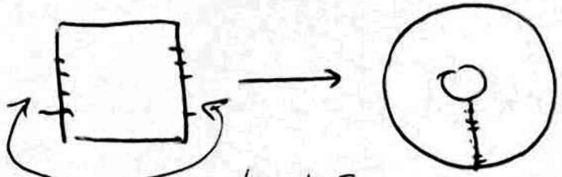
i) $f_n b_i = 0 = b_i f_n$, $\quad 0 \le i \le n-1$.

ii) f_{n-1} belongs to subalgebra generated by $\{b_0, \dots, b_{n-1}\}$.

iii) $\tilde{f}_n = e_n \in \mathcal{B}$

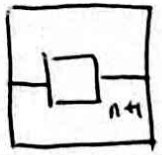
"close up" map

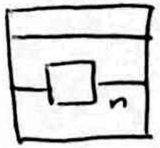
$\sim : TL_n \longrightarrow B$ induced by

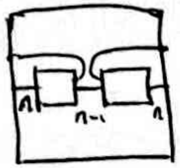


glue up together

• Recurrence formula: $f_1 = 1$ and



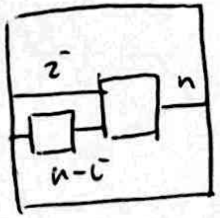
$=$


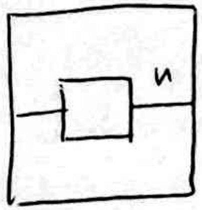
$- \dfrac{\langle e_{n-1}\rangle}{\langle e_n\rangle}$


$(*)$.

Lemma:

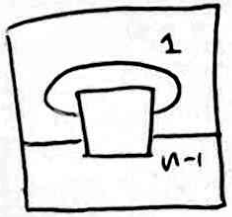
1) (贪吃蛇)

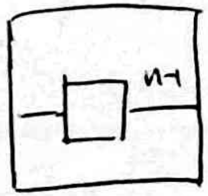


$=$


, $0 \le i \le n$ $(**)$

2)



$= \dfrac{\langle e_n\rangle}{\langle e_{n-1}\rangle}$


$(***)$

Thm (Reshetikhin-Turaev)

$M = M_L$, D standard diagram, $b_\pm$ number of $+/-$ eigenvalues of linking matrix of L.

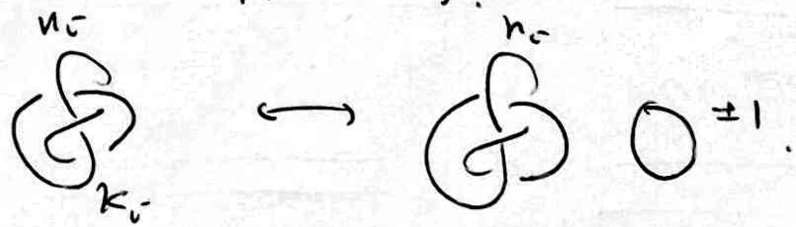
Then $\forall\, r \ge 3$,

$$RT_r(M) = \langle \omega_r, \dots, \omega_r\rangle_D \, \langle \omega_r\rangle_{U_+}^{-b_+} \, \langle \omega_r\rangle_{U_-}^{-b_-}$$

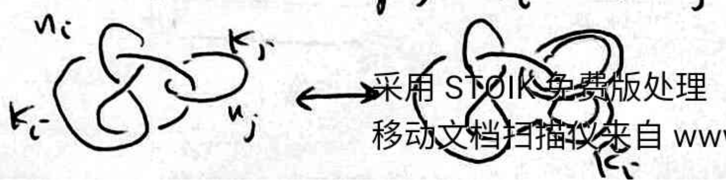
is invariant under KM I & KM II.

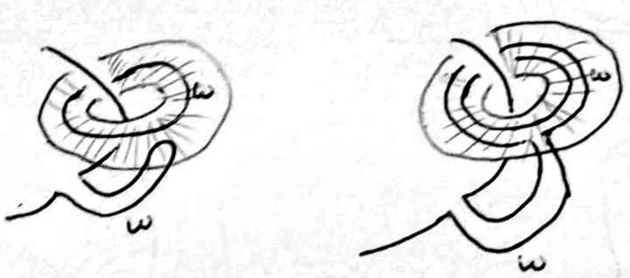
• $\omega_r = \sum_{n=0}^{r-2} \langle e_n\rangle e_n \in B_r$.

• KM I (blow up/down).

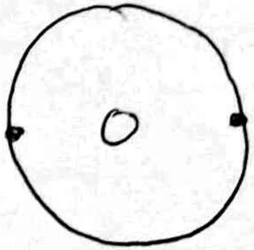


• KM II (handle sliding) $\tilde{n}_i = n_i + n_j \pm 2 lk_{ij}$.





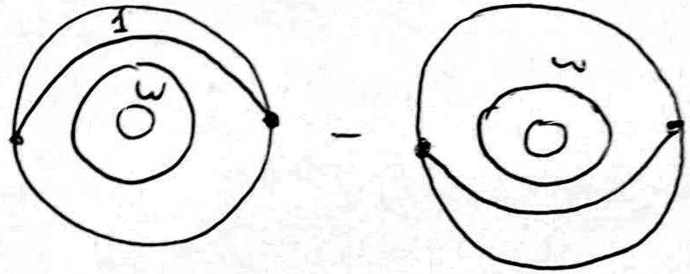
Consider diagrams in



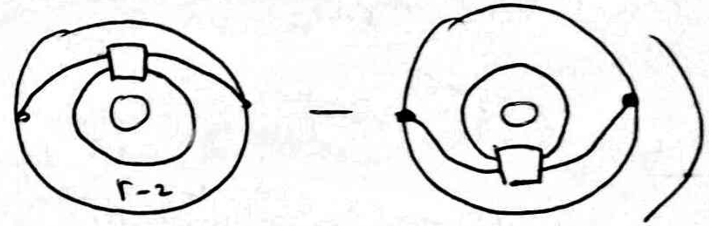
modulo ① $\times = A\,)(+ A^{-1} \asymp$

② $\bigcirc = -A^{2} - A^{-2}$.

lemma 1: each term is a linear combination



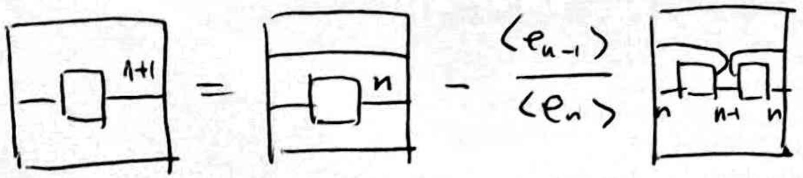
$= \pm \Big($



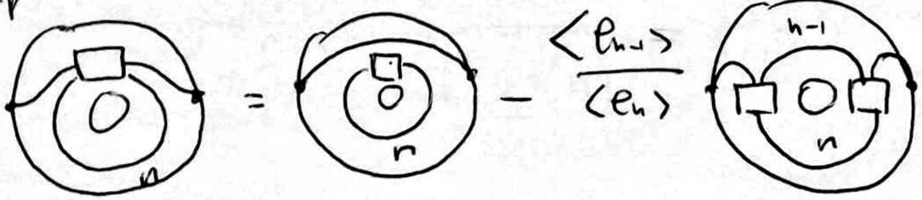
$\Big)$

---

pf: By (*),

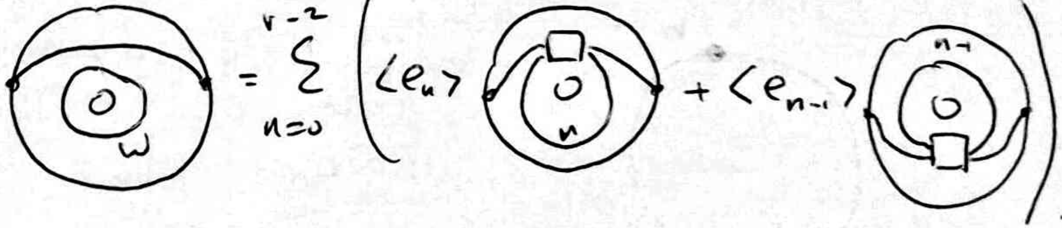


close up n pts ⇒

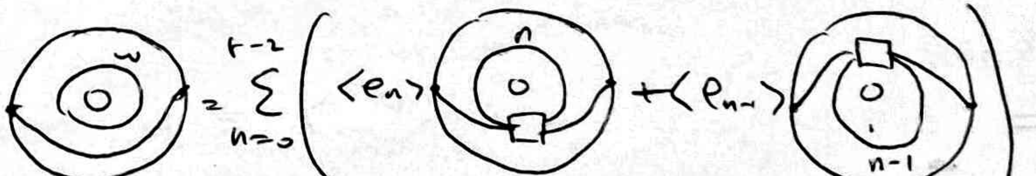


(**) ⇒

⇒ (I)



Rotating by 90°, ⇒ (II)



$(I)-(II) = \underbrace{\langle e_{r-2}\rangle}_{\shortparallel\ (-1)^{r-2}} \Big($

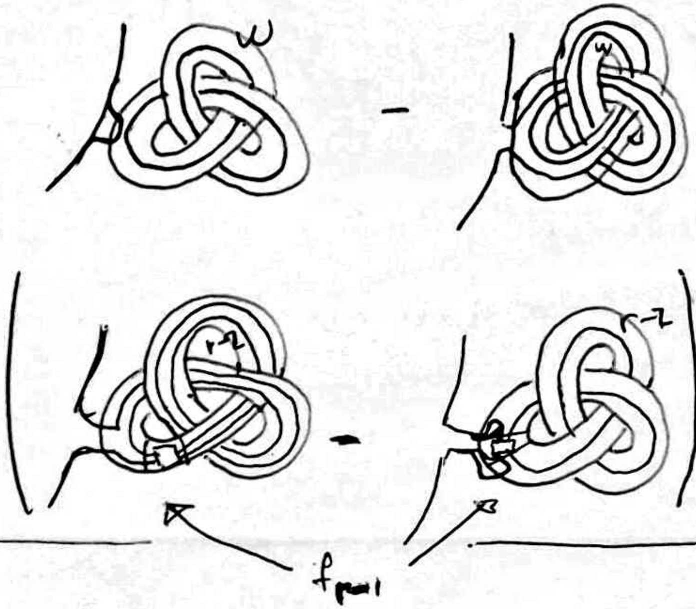
Cor 1. If D and D' are two Kirby diagrams differed by a handle sliding of some component over the component K_1, then

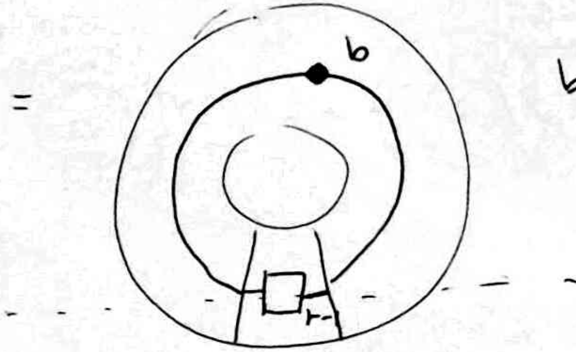
$$\langle \omega_r, -, \cdots, - \rangle_D = \langle \omega_r, -, \cdots, - \rangle_{D'}$$

as multilinear map on $\mathcal{B}_r$.

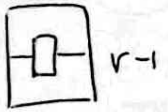
Cor 2. $\langle \omega_r, \ldots, \omega_r \rangle_D$ is invariant under KM II.

Pf of Cor 2: By Lemma 1,

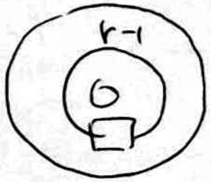




by pushing everything above



[illegible],

$= c \cdot$  $= c \cdot \langle e_{r-1} \rangle = 0.$

constant

□

Remains to show that $\langle \omega_r \rangle_{u_\pm} \neq 0$.

Lemma 2:

1) $\langle \omega_r \rangle = \dfrac{-2r}{(A^2 - A^{-2})^2} \neq 0.$

2) $\langle \omega_r \rangle_{u_+} \langle \omega_r \rangle_{u_-} = \langle \omega_r \rangle$

Pf(1):

$$\langle \omega_r \rangle = \sum_{n=0}^{r-2} \langle e_n \rangle^2 = \sum_{n=0}^{r-2} (-1)^{2n} \left(\frac{A^{2n+2} - A^{-2n-2}}{A^2 - A^{-2}} \right)^2$$

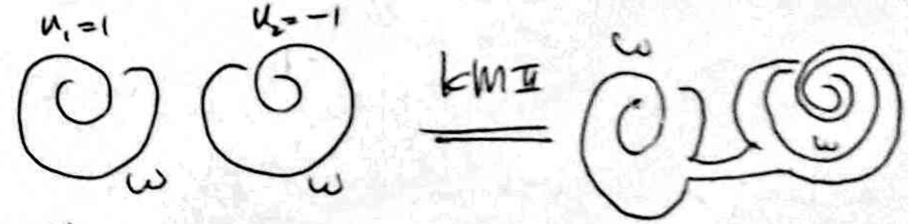
$$= \frac{1}{(A^2-A^{-2})^2} \left(-2r + \left(\sum_{n=0}^{r-2} A^{4n+4} + 1 \right) + \left(\sum_{n=0}^{r-2} A^{-4n-4} + 1 \right) \right)$$

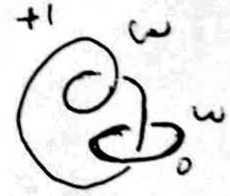
$$= \frac{1}{(A^2-A^{-2})^2} \left(-2r + \overset{=0}{\cancel{\frac{A^{4r}-1}{A^4-1}}} + \overset{=0}{\cancel{\frac{A^{-4r}-1}{A^4-1}}} \right)$$

$$= \frac{-2r}{(A^2-A^{-2})^2}.$$

For (2), we ~~need~~ have

$\langle \omega_r \rangle_{u_+} \langle \omega_r \rangle_{u_-}$





$\tilde{n}_2 = n_1 + n_2 = 0$.

$\tilde{lk}_{12} = lk_{12} \pm n_1 = \pm 1$.

want $\overset{?}{=}$

Recall. $t e_n = (-1)^n A^{n^2+2n} e_n$. Then

$$= \sum_{n=0}^{r-2} \langle e_n \rangle (-1)^n A^{n^2+2n}$$

$= t$

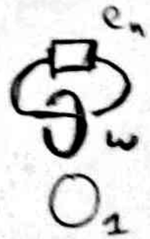
want $\overset{?}{=}$

Also: $c e_n = (-A^{2n+2} - A^{-2n-2}) e_n$

For $n > 0$.

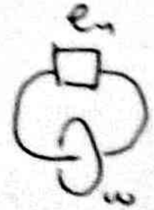
$(-A^2 - A^{-2})$

$=$



$\overset{\text{KMII}}{\underset{\text{(or Comm1)}}{=}}$

$= (-A^{2n+2} - A^{-2n-2})$



Since $-A^2 - A^{-2} \neq -A^{2n+2} - A^{-2n-2}$ for $0 < n \le r-2$,

$= 0$.

<u>Normalization</u>:

Let μ s.t. $\mu^{-2} = \langle\omega_r\rangle_{u_+}\langle\omega_r\rangle_{u_-}$, i.e.,

$$\langle\mu\omega_r\rangle_{u_+} = \langle\mu\omega_r\rangle_{u_-}^{-1}. \quad \text{and}$$

$$\langle\mu\omega_r\rangle = \mu^{-1}.$$

In fact,

$$\mu = \frac{A^2 - A^{-2}}{\sqrt{-2r}}.$$

<u>Def</u>: Let σ be the signature of the linking matrix of L, i.e.

$$\sigma = b_+ - b_-.$$

Define

$$I_r(M) = \frac{\langle\mu\omega_r, \dots, \mu\omega_r\rangle_D \langle\mu\omega_r\rangle_{u_-}^{\sigma}}{\langle\mu\omega_r\rangle}$$

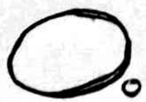
$$= \boxed{\mu\langle\mu\omega_r, \dots, \mu\omega_r\rangle_D \langle\mu\omega_r\rangle_{u_-}^{\sigma}}$$

Eg.:

$$I_r(S^3) = \mu.$$

$$I_r(S^1\times S^2) = \mu\langle\mu\omega_r\rangle = 1.$$

$$S^1\times S^2 = S^3_{u_0}$$



Why? (HW).

Rm:

Everything works for A primitive $2r$-th root of 1, r odd.

Only difference is

Lemma 2, 2).

$$\langle\omega_r\rangle_{u_+}\langle\omega_r\rangle_{u_-} = 2\langle\omega_r\rangle$$ (HW)

Lecture 6

Last time:

$$\mu^{-2} = \langle\omega\rangle_{u_+}\langle\omega\rangle_{u_-} = \langle\omega\rangle = \sum_{n=0}^{r-2}\langle e_n\rangle^2 = \frac{-2r}{(A^2-A^{-2})^2}$$

Normalized RT invariant

$$\boxed{I_r(M) = \mu\cdot\langle\mu\omega,\dots,\mu\omega\rangle_D\,\langle\mu\omega\rangle_{u_-}^{\sigma}}$$

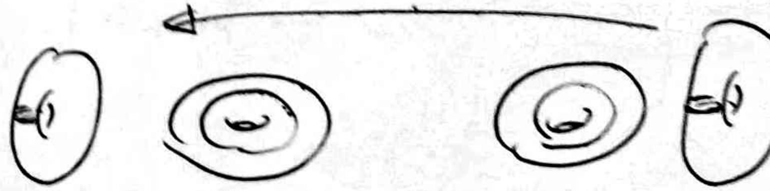
Since $\langle\mu\omega\rangle_{u_+} = \langle\mu\omega\rangle_{u_-}^{-1}$,

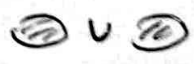
$$\langle\mu\omega\rangle_{u_-}^{\sigma} = \langle\mu\omega\rangle_{u_-}^{b_+-b_-} = \langle\mu\omega\rangle_{u_+}^{-b_+}\langle\mu\omega\rangle_{u_-}^{-b_-}.$$

Eg. $I_r(S^3) = \mu = \dfrac{A^2-A^{-2}}{\sqrt{-2r}}$.

$I_r(S^2\times S^1) = 1$.

$S^2\times S^1 = M_{u_0}$ $\quad \mu\cdot\langle\mu\omega\rangle = 1$.

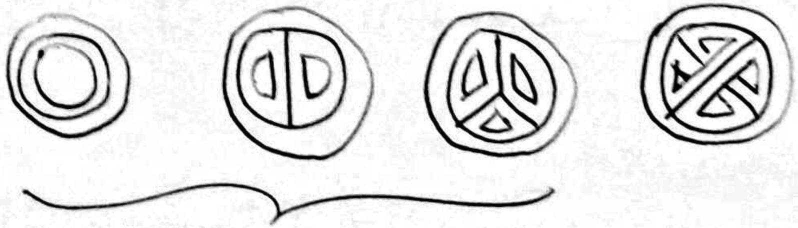


 $= S^2$. $\quad \underline{\underline{S^2\times S^1}}$.

Turaev – Viro invariants

Def. A ribbon graph $\Gamma \subset \mathbb{R}^3$ is a 3-valent graph w/ a 2-dimensional thickening.

Eg.



only need these 3.

• $r \geq 3$. $J_r = \{0, 1, \dots, r-2\}$.

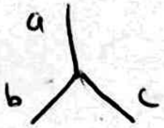
Def. An r-admissible coloring of Γ is an assignment of an element of J_r to each edge of Γ s.t. $\forall$ vertex of Γ

(i) $a+b \geq c$, $b+c \geq a$, $c+a \geq b$

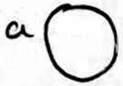
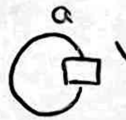
(ii) $a+b+c \leq 2(r-2)$

(iii) $a+b+c$ …

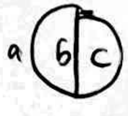
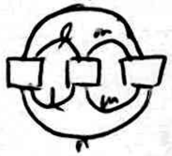
Each r-adm coloring c determines a complex number $\langle \Gamma, c \rangle$ as follows

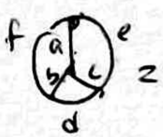
- edge  Jones-Wenzl.
- vertex  $\longmapsto$  $l = \frac{a+b-c}{2}$, $m = \frac{b+c-a}{2}$, $n = \frac{a+c-b}{2}$
- take the Kauffman bracket (A = primitive $4r$-th root of 1)

Eg:

 $\langle$  $\rangle = \langle e_a \rangle = (-1)^a \dfrac{A^{2a+2} - A^{-2a-2}}{A^2 - A^{-2}}$.

- $[n] = \dfrac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}$, $[n]! = [n] \cdots [1]$, $[0]! = 1$

 =  $= (-1)^{\frac{a+b+c}{2}} \dfrac{\left[\frac{a+b+c}{2}+1\right]! \left[\frac{a+b-c}{2}\right]! \left[\frac{b+c-a}{2}\right]! \left[\frac{a+c-b}{2}\right]!}{[a]![b]![c]!}$

 $= \dfrac{\prod_{i=1}^{4} \prod_{j=1}^{3} [Q_j - T_i]!}{[a]![b]! \cdots [f]!} \displaystyle\sum_{k=\max\{T_i\}}^{\min\{Q_j\}} \frac{(-1)^k [k+1]!}{\prod_{i=1}^{4} [k-T_i]! \prod_{j=1}^{3} [Q_j - k]!}$

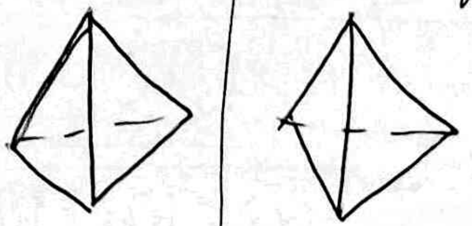
$T_1 = \frac{a+b+c}{2}$, $T_2 = \frac{a+e+f}{2}$, $T_3 = \frac{b+d+f}{2}$, $T_4 = \frac{c+d+e}{2}$

$Q_1 = \frac{a+b+d+e}{2}$, $Q_2 = \frac{a+c+d+f}{2}$, $Q_3 = \frac{b+c+e+f}{2}$.

- M closed 3-mfd.
- A triangulation of M consists of a finite collection of Euclidean tetrahedra $\sigma_1, \ldots, \sigma_n$ and a set of homeomorphisms $\{\phi_{ij}\}$ between pairs of faces of $\{\sigma_k\}$ s.t.

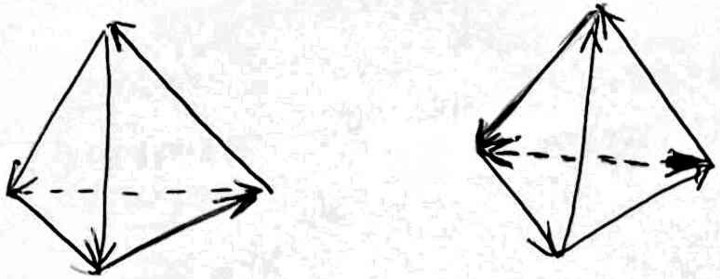
$$M = \bigsqcup_{k=1}^{n} \sigma_k \Big/ \{\phi_{ij}\}$$

Eg. S^3.



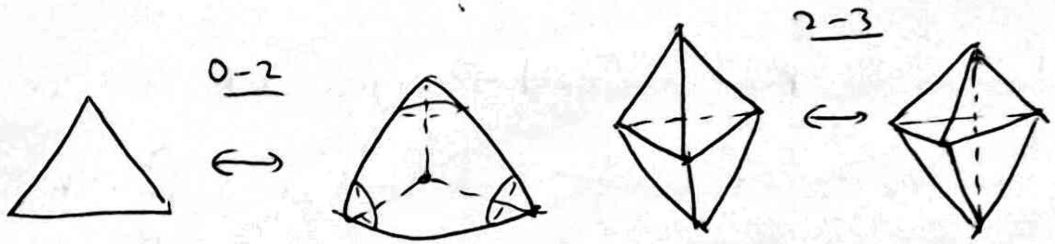
mirror map.

$S^2 \times S^1$



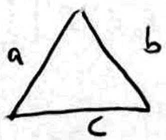
Thm (Matveev, Piergallini)

Any two triangulations of M are related by a sequence of 0–2 and 2–3 Pachner Moves



- $(M, \mathcal{T})$ V, E, F, T sets of vertices, edges, faces and tetrahedra of $\mathcal{T}$.

A coloring $c : E \to I_r$ is r-admissible if for each $f \in F$



(a, b, c) is r-admissible, i.e.

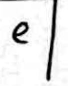
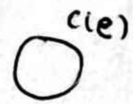
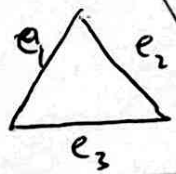
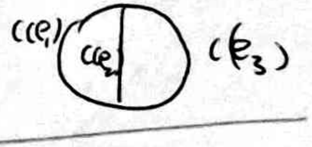
(i) $a+b \geq c$, $b+c \geq a$, $c+a \geq b$

(ii) $a+b+c \leq 2(r-2)$

(iii) $a+b+c$ even.

$\mathcal{A}_r = \mathcal{A}_r(M, \mathcal{T}) = \{ r\text{-adm colorings} \}$.

For $c \in \mathcal{A}_r$, ~~let~~

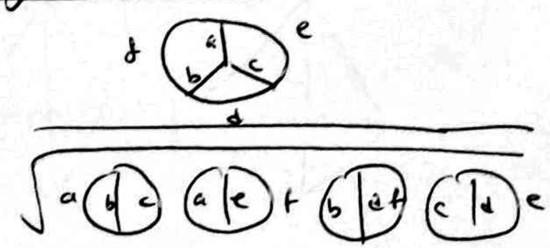
for	let	dual picture
	$\lvert e \rvert_c =$ 	
$f =$  ,	$\lvert f \rvert_c =$ 	
$\sigma =$  ,	$\lvert \sigma \rvert_c$ 	

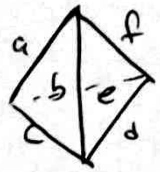
Ref/ Thm (Turaev–Viro). For $r \geq 3$, let $\eta = \dfrac{-2r}{(A^2 - A^{-2})^2}$. Let T be a triangulation of M. Then

$$TV_r(M) = \eta^{-|V|} \sum_{c \in \mathcal{A}_r} \prod_{e \in E} |e|_c \prod_{f \in F} |f|_c^{-1} \prod_{\sigma \in T} |\sigma|_c$$

defines a real valued invariant of M, i.e., is invariant under 0-2 and 2-3 Pachner moves.

quantum 6j-symbol

$\left|\begin{matrix} a & b & c \\ d & e & f \end{matrix}\right| \doteq$ 

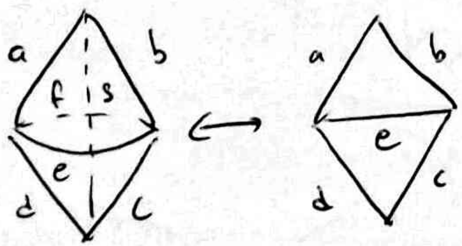
For $\sigma =$ , let $\|\sigma\|_c = \left|\begin{matrix} a & b & c \\ d & e & f \end{matrix}\right|$. Then

$$TV_r(M) = \eta^{-|V|} \sum_{c \in \mathcal{A}_r} \prod_{e \in E} |e|_c \prod_{\sigma \in T} \|\sigma\|_c .$$

Orthogonality. If (a,b,e), (a,b,f), (c,d,e), (c,d,f) admissible, then

$$\sum_s |s||e| \left|\begin{matrix} a & b & e \\ c & d & s \end{matrix}\right| \left|\begin{matrix} a & b & f \\ c & d & s \end{matrix}\right| = \delta_{ef},$$

where $s \in I_r$ s.t. $(a,d,s)(b,c,s)$ admissible.

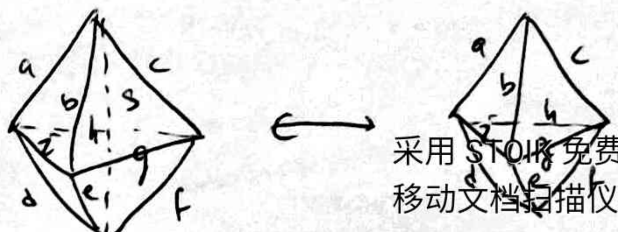


$|a| = (-1)^a [a+1] = \langle e_a \rangle.$

Biedenharn–Elliott identity: If (a,b,i), (b,c,g), (a,c,h), (d,e,i) (e,f,g) (d,f,h) adm, then

$$\sum_s |s| \left|\begin{matrix} a & b & i \\ e & d & s \end{matrix}\right| \left|\begin{matrix} b & c & g \\ f & e & s \end{matrix}\right| \left|\begin{matrix} c & a & h \\ d & f & s \end{matrix}\right| = \left|\begin{matrix} a & b & i \\ g & h & c \end{matrix}\right| \left|\begin{matrix} d & e & i \\ g & h & c \end{matrix}\right|,$$

where $s \in I_r$ s.t. (a,d,s) (b,e,s) (c,f,s) adm.



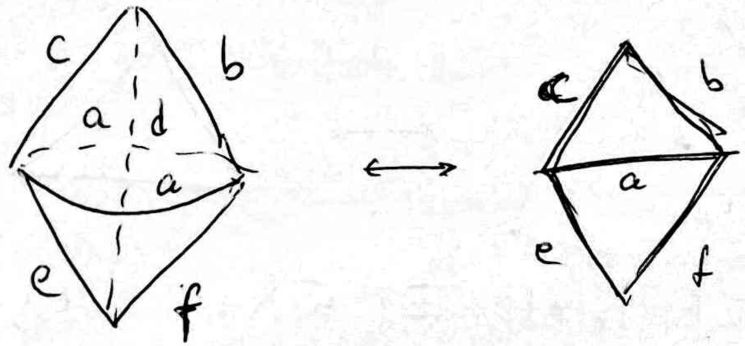
Cor of Orthogonality. If (a,b,c) admissible, then

$$\sum_{d,e,f} |d||e||f| \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = \eta$$

where $d,e,f \in I_r$ s.t. (a,e,f) (b,d,f), (c,d,e) admissible.

pf. By Orthogonality,

$$\sum_{d} |d||a| \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = 1.$$



Therefore,

$$\text{LHS} = \eta^{-1} \sum_{e,f} |a|^{-1}|e||f|, \text{ where}$$

$e,f \in I_r$ s.t. (a,e,f) adm.

If $a=0$, then

$$\sum_{e,f} |a|^{-1}|e||f| = \sum_{e=0}^{r-2} |e|^2 = \eta.$$

By Lemma 1, done.

Lemma 1. $\forall$ $a,b \in I_r$,

$$\sum_{e,f} |a|^{-1}|e||f| = \sum_{g,h} |b|^{-1}|g||h|,$$

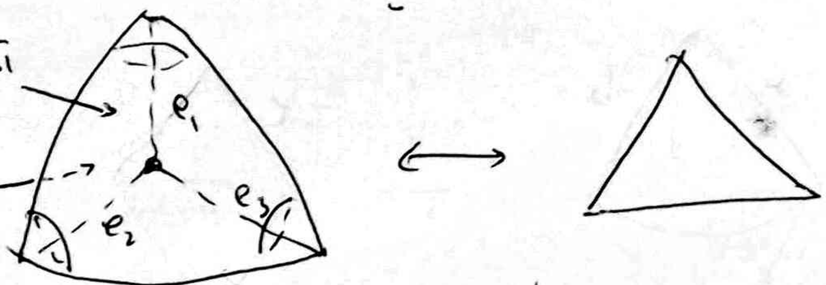
where (a,e,f), (b,g,h) admissible.

pf: Direct calculation for $b=a+1$, then by induction.

pf: Invariance of 0-2 move.

Suppose $\mathcal{T}'$ is from $\mathcal{T}$ by a 0-2 move.

Let V', E', T' be sets of vertices, edges / tetra of $\mathcal{T}'$.



Then $|V'| = |V|+1$, $E = E'\setminus\{e_1, e_2, e_3\}$ and $T = T'\setminus\{\sigma_1, \sigma_2\}$.

$$TV_r(M,\mathcal{T}') = \mu^{-|V'|}\sum_{c\in A_r(\mathcal{T}')}\prod_{e\in E'}|e|_c\prod_{\sigma\in T'}\|\sigma\|_c$$

$$= \mu^{-|V'|}\sum_{c\in A_r(\mathcal{T}')}\prod_{e\in E'\setminus\{e_1,e_2,e_3\}}|e|_c\prod_{\sigma\in T'\setminus\{\sigma_1,\sigma_2\}}\|\sigma\|_c\left(\sum_{d,e,f}|d||e||f|\|\sigma_1\|_c\|\sigma_2\|_c\right)$$

$$\overset{\text{Cor}}{=} \mu^{-|V'|+1}\sum_{c\in A_r(\mathcal{T}')}\prod_{e\in E'\setminus\{e_1,e_2,e_3\}}|e|_c\prod_{\sigma\in T'\setminus\{\sigma_1,\sigma_2\}}\|\sigma\|_c$$

$$= \mu^{-|V|}\sum_{c\in A_r(\mathcal{T})}\prod_{e\in E}|e|_c\prod_{\sigma\in T}\|\sigma\|_c = TV_r(M,\mathcal{T})$$

Invariance under 2-3 is similar (HW).

Eg 2: $TV_r(S^1\times S^2) = 1$

Eg 1: $TV_r(S^3) = \eta^{-1} = \dfrac{(A^2-A^{-2})^2}{-2r}$.

$$TV_r(S^3) = \eta^{-4}\sum_{a,b,c,d,e,f}|a|\cdots|f|\begin{vmatrix}a&b&c\\d&e&f\end{vmatrix}\begin{vmatrix}a&b&c\\d&e&f\end{vmatrix}$$

$$\overset{\text{Cor}}{=} \eta^{-3}\sum_{a,b,c}|a||b||c|$$

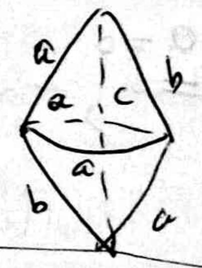
$$= \eta^{-3}\sum_a|a|^2\left(|a|^{-1}\sum_{b,c}|b||c|\right)$$

$$\overset{\text{Lemma 1}}{=} \eta^{-2}\sum_a|a|^2 = \eta^{-1}.$$

$$TV_r(S^1\times S^2) = \eta^{-1}\sum_{a,b,c}|a||b||c|\begin{vmatrix}a&a&b\\a&c&b\end{vmatrix}\begin{vmatrix}a&a&b\\a&c&b\end{vmatrix}$$

$$\overset{\text{Orthogonality}}{=} \eta^{-1}\sum_{a,b}|b|$$

$$\overset{\text{direct calculation (HW)}}{=} 1.$$



Observation: $\boxed{\begin{gathered}TV_r(S^3) = |\tau_r(S^3)|^2\\ TV_r(S^1\times S^2) = |\tau_r(S^1\times S^2)|^2\end{gathered}}$

lecture 7

last time:

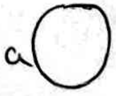
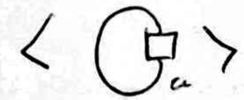
$r \geq 3$, $I_r = \{0, 1, \dots, r-2\}$. A primitive $4r$-th root of 1.

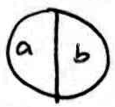
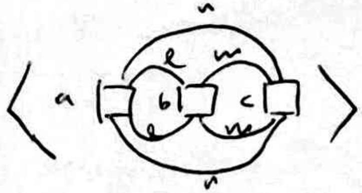
- $(a, b, c) \in I_r^3$ is r-admissible if

(i) $a+b \geq c$, $b+c \geq a$, $c+a \geq b$

(ii) $a+b+c \leq 2(r-2)$,

(iii) $a+b+c$ is even.

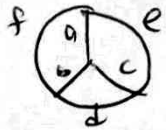
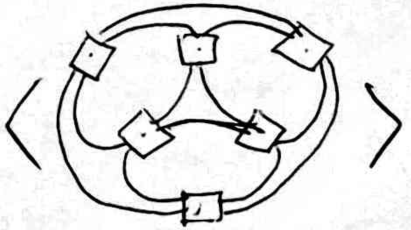
 $=\langle$  $\rangle = (-1)^a [a+1]$

 $c = \langle$  $\rangle$

$l = \frac{a+b-c}{2}$

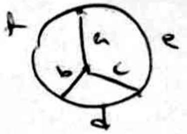
$m = \frac{b+c-a}{2}$

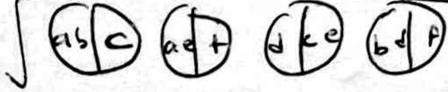
$n = \frac{a+c-b}{2}$

 $= \langle$  $\rangle$. symmetric in abc.

quantum 6j-symbols

$$\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = \frac{\text{(tetrahedron graph)}}{\sqrt{\cdots}}$$





Orthogonality: $\sum_s O_s O_e \begin{vmatrix} a & b & e \\ c & d & s \end{vmatrix} \begin{vmatrix} a & b & f \\ c & d & s \end{vmatrix} = \delta_{ef}$

BE identity

$$\sum_s O_s \begin{vmatrix} a & b & i \\ e & d & s \end{vmatrix} \begin{vmatrix} b & c & g \\ f & e & s \end{vmatrix} \begin{vmatrix} c & a & h \\ d & f & s \end{vmatrix} = \begin{vmatrix} a & b & i \\ g & h & c \end{vmatrix} \begin{vmatrix} d & e & i \\ g & h & f \end{vmatrix}$$

Def/Thm (Turaev-Viro)

(M, T) triangulated 3-mfd, closed. V, E, F, T sets of vertices, edges, faces, tetra. $\eta = \frac{-2r}{(A^2 - A^{-2})^2}$. Then

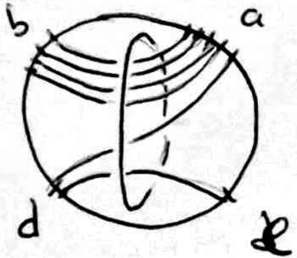
$$TV_r(M) = \eta^{-|V|} \sum_{c \in \mathcal{A}_r} \prod_{e \in E} O_e \prod_{f \in F} \text{(theta)}^{-1} \prod_{\sigma \in T} \text{(tetrahedron)}$$

defines a real valued invariant of M

Goal today. Pf Orthogonality and BE.

Let D_{abde} be $\mathbb{D}^2$ w/ $a+b+d+e$ pts on $\partial\mathbb{D}^2$.

Let $a, b, d, e \in I_r$ s.t. $a+b+d+e$ even

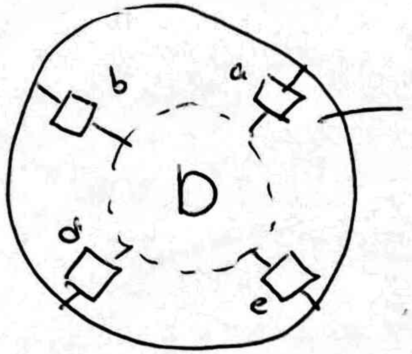


Let $K_A(D_{abde})$ be $\mathbb{Z}[A^{\pm}]$-mod generated by link diagrams in D_{abde} modulo

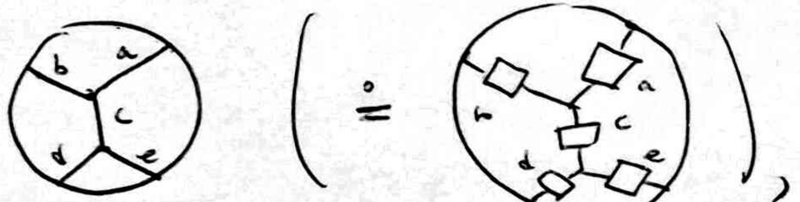
① $\times = A\,)(+ A^{-1} \asymp$

② $\bigcirc = -A^2 - A^{-2}$.

Let T_{abde} be the sub-module generated by the diagrams of the form

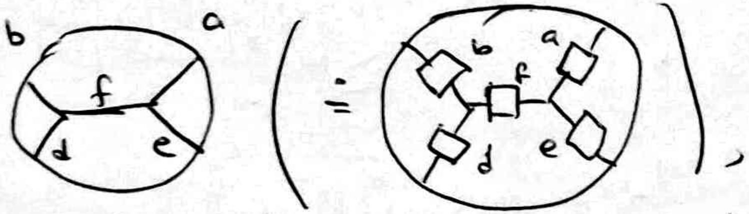


Eg. (i)

$Y_c =$ 

where $c \in I_r$ s.t. (a,b,c), (c,d,e) admissible.

(ii)

$Y'_f =$ 

where $f \in I_r$ s.t. (b,d,f), (a,e,f) admissible

Prop: $\{Y_c\}_c$ and $\{Y'_f\}_f$ are basis of T_{abde}.

Pf. later.

Let $B_c = \sqrt{\dfrac{O_c}{(a|b|c)(c|d|e)}} \cdot Y_c$, $B'_f = \sqrt{\dfrac{O_f}{(b|d|f)(a|e|f)}}\, Y'_f$.

Cor: $\{B_c\}_c$ and $\{B'_c\}_c$ are basis of T_{abde}

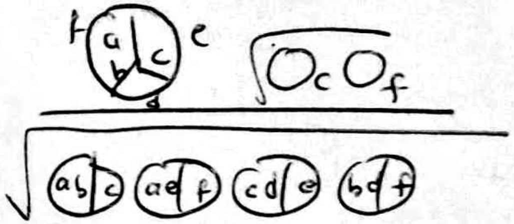
Def: let $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \in \mathbb{Z}[A^{\pm}] \subset \mathbb{C}$ be the unique complex number s.t

$$B_c = \sum_f \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} B'_f ,$$

where $f \in I_r$ s.t (b,d,f), (a,e,f) adm.

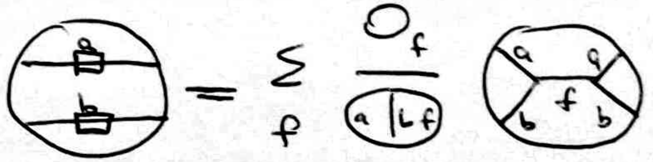
Def: $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$ is the renormalized 6j-symbol.

Prop 2:

(i) $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} =$ 

(ii) $\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = \dfrac{1}{\sqrt{O_c O_f}} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$.

Pf. later.

Cor: 

(Need it for the pf of $TV_r = |I_H|^2$)

Prop 3:

(i) $\sum_s \begin{Bmatrix} a & b & e \\ c & d & s \end{Bmatrix} \begin{Bmatrix} a & b & f \\ c & d & s \end{Bmatrix} = \delta_{ef}$

(ii) $\cancel{\sum_s \begin{Bmatrix} a & b & c \\ e & d & s \end{Bmatrix} \begin{Bmatrix} b & c & g \\ f & e & s \end{Bmatrix} \begin{Bmatrix} c & a & h \\ d & f & s \end{Bmatrix} = \begin{Bmatrix} a & b & i \\ g & h & c \end{Bmatrix} \begin{Bmatrix} d & e & i \\ g & h & f \end{Bmatrix}}$

$$\sum_s \begin{Bmatrix} c & b & g \\ e & f & s \end{Bmatrix} \begin{Bmatrix} h & a & c \\ s & f & d \end{Bmatrix} \begin{Bmatrix} a & b & i \\ e & d & s \end{Bmatrix} = \begin{Bmatrix} h & a & c \\ b & g & i \end{Bmatrix} \begin{Bmatrix} h & i & g \\ e & f & d \end{Bmatrix}$$

Cor

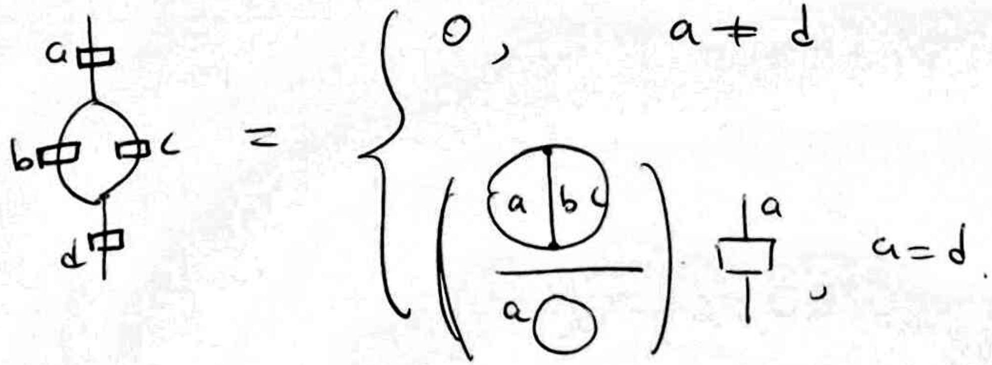
Orthogonality:

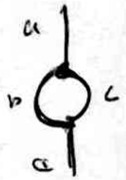
$$\sum_s O_s O_e \begin{vmatrix} a & b & e \\ c & d & s \end{vmatrix} \begin{vmatrix} a & b & f \\ c & d & s \end{vmatrix} = \delta_{ef}$$

BE identity:

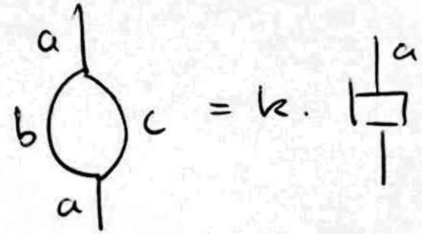
$$\sum_s O_s \begin{vmatrix} a & b & c \\ e & d & s \end{vmatrix} \begin{vmatrix} b & c & g \\ f & e & s \end{vmatrix} \begin{vmatrix} c & a & h \\ d & f & s \end{vmatrix} = \begin{vmatrix} a & b & i \\ g & h & c \end{vmatrix} \begin{vmatrix} d & e & i \\ g & h & f \end{vmatrix}$$

Lemma:

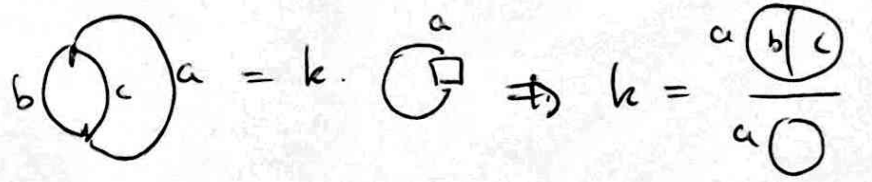




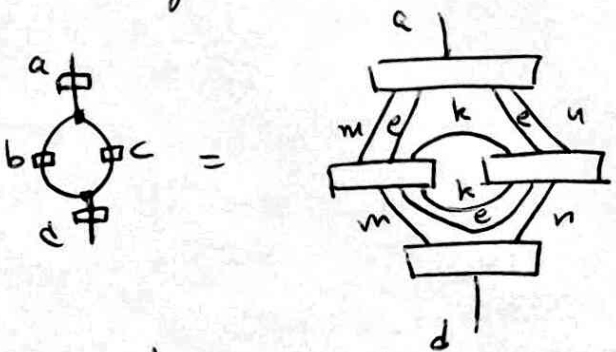
pf. If $a=d$, then



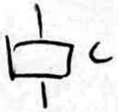
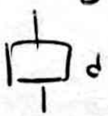
for some $k \in \mathbb{Z}[A^{\pm 1}]$.

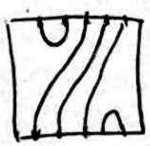


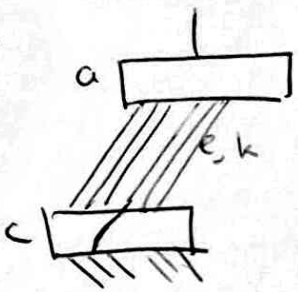
If w.l.o.g $a>d$, then

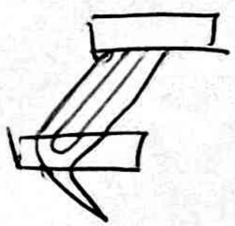


, where $e = \frac{a-d}{2}$.

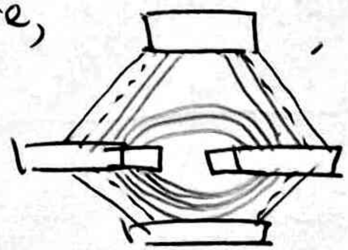
Each  or  is a linear combination of embedded diagrams



If 

, then there must be a "turn back" , since there are more "in" than "out".

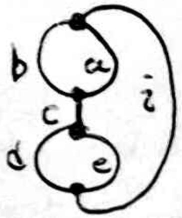
⇒ "= 0".

Therefore, , which contains a "turn back", since there is no "out"

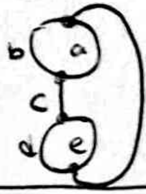
⇒ "= 0".

Pf of Prop 1.2: • $\{Y_c\}$ are linearly independent.

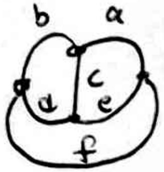
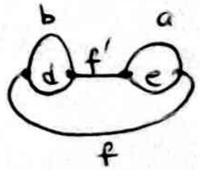
Suppose $\sum_{c=0}^{n} k_c Y_c = 0$, then $\forall\, i \in I_r$,

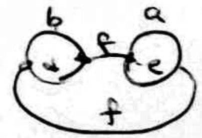
$\sum_{k_c} k_c \cdot$  $= 0$.

Lemma $\Rightarrow$

$0 = k_c \cdot$  $= k_c \dfrac{\theta(a,b,c)\,\theta(c,d,e)}{O_c} \Rightarrow k_c = 0.$

• Prop 2(i)

$\sqrt{\dfrac{O_c}{\theta(a,b,c)\,\theta(c,d,e)}}$  $= \sum_{f'} \begin{Bmatrix} a & b & c \\ d & e & f' \end{Bmatrix} \sqrt{\dfrac{O_{f'}}{\theta(b,d,f')\,\theta(a,e,f')}}$ 

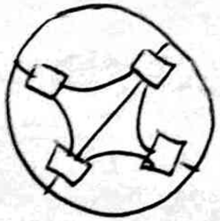
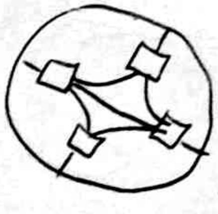
$\overset{\text{Lemma}}{=} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} \sqrt{\dfrac{O_f}{\theta(b,d,f)\,\theta(a,e,f)}}$ 

$\Rightarrow \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \sqrt{\dfrac{O_c}{\theta(a,b,c)\,\theta(c,d,e)}}$  $\sqrt{\dfrac{\theta(b,d,f)\,\theta(a,e,f)}{O_f}} \cdot \dfrac{O_f}{\theta(b,d,f)\,\theta(a,e,f)}$

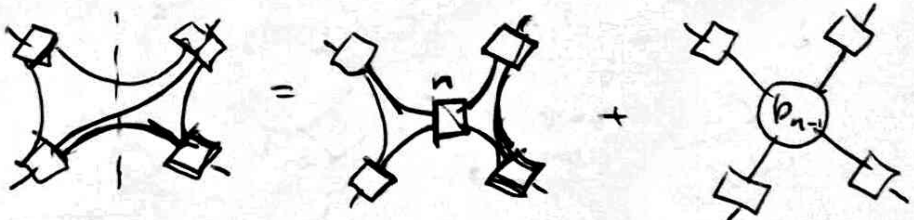
$= \dfrac{\text{(tet)}\sqrt{O_c O_f}}{\sqrt{\theta(a,b,c)\,\theta(c,d,e)\,\theta(b,d,f)\,\theta(a,e,f)}}$ 

• $\{Y_c\}$ span T_{abde}.

Each element of T_{abde} is a linear combination of embedded diagrams, which is non-zero iff

 or , since otherwise there is a "turn back".

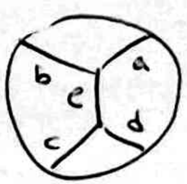
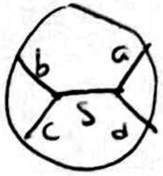
Since 1_n  $+ D_{n-1}$, where D_{n-1} has "turn backs",

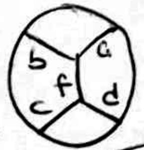


where D_{n-1} has less than n intersections with ⋮.

Induction $\Rightarrow$.

Pf of Prop3 :

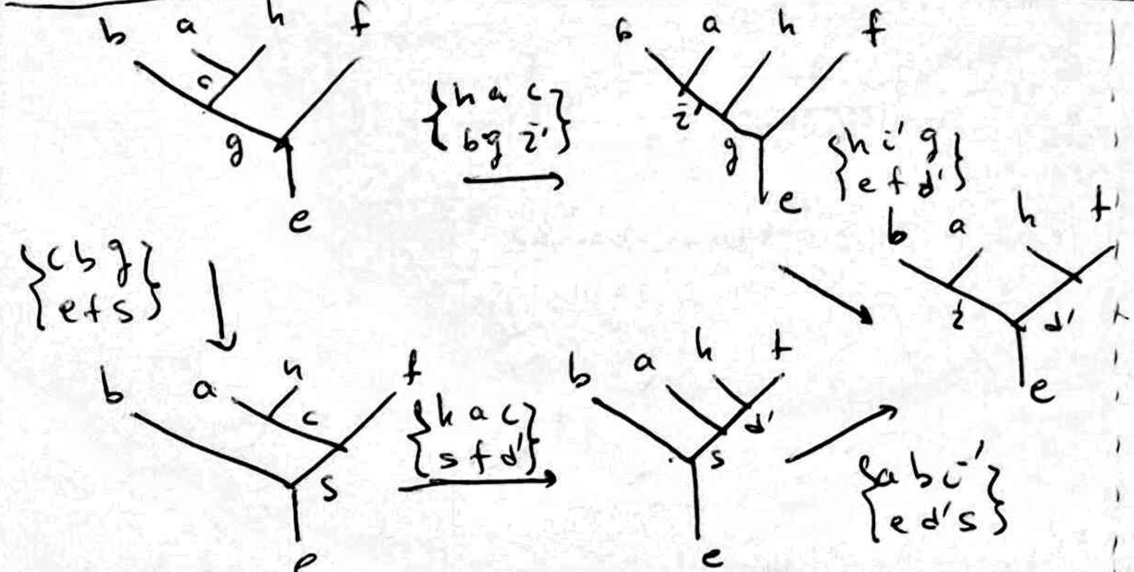
$\sqrt{\dfrac{O_e}{\theta(abe)\,\theta(cde)}}$  $= \sum_s \begin{Bmatrix} a & b & e \\ c & d & s \end{Bmatrix} \sqrt{\dfrac{O_s}{\theta(ads)\,\theta(bcs)}}$ 

$= \sum_f \sum_s \begin{Bmatrix} a & b & e \\ c & d & s \end{Bmatrix} \begin{Bmatrix} a & b & f \\ c & d & s \end{Bmatrix} \sqrt{\dfrac{O_f}{\theta(adf)\,\theta(bcf)}}$ 

$\Rightarrow \sum_s \begin{Bmatrix} a & b & e \\ c & d & s \end{Bmatrix} \begin{Bmatrix} a & b & f \\ c & d & e \end{Bmatrix} = \delta_{ef}$.

Similar to pf of Prop2 (i).

$$B'_f = \sum_c \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} B_c .$$



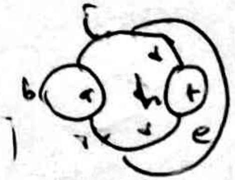
For i, d have

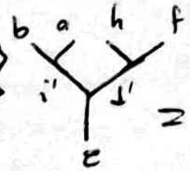
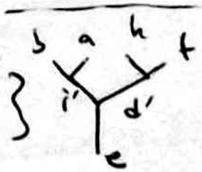
$\sum_{i'} \sum_{d'} \left(\begin{Bmatrix} h & a & c \\ b & g & i' \end{Bmatrix} \begin{Bmatrix} h & i' & g \\ e & f & d' \end{Bmatrix} - \sum_s \begin{Bmatrix} c & b & g \\ e & f & s \end{Bmatrix} \begin{Bmatrix} h & a & c \\ s & f & d \end{Bmatrix} \begin{Bmatrix} a & b & i' \\ e & d & s \end{Bmatrix} \right)$  $= 0$

By lemma :

$\left(\begin{Bmatrix} h & a & c \\ b & g & i \end{Bmatrix} \begin{Bmatrix} h & i & g \\ e & f & d \end{Bmatrix} - \sum_s \begin{Bmatrix} c & b & g \\ e & f & s \end{Bmatrix} \begin{Bmatrix} h & a & c \\ s & f & d \end{Bmatrix} \begin{Bmatrix} a & b & i \\ e & d & s \end{Bmatrix} \right)$  $= 0.$

By Lemma (iii) :

 $= \dfrac{\theta(afi)\,\theta(dhf)\,\theta(ide)}{O\,O_d} \neq 0$

$\Rightarrow \sum_{i'} \sum_{d'} \begin{Bmatrix} h & a & c \\ b & g & i' \end{Bmatrix} \begin{Bmatrix} h & i' & g \\ e & f & d' \end{Bmatrix}$  $= \sum_{i'} \sum_{d'} \sum_s \begin{Bmatrix} c & b & g \\ e & f & s \end{Bmatrix} \begin{Bmatrix} h & a & c \\ s & f & d' \end{Bmatrix} \begin{Bmatrix} a & b & i' \\ e & d' & s \end{Bmatrix}$ 

Goal Today: Lecture 8

Thm (Turaev, Walker, Roberts)
M closed orientable 3-mfd. Then

$$TV_r(M) = |I_r(M)|^2.$$

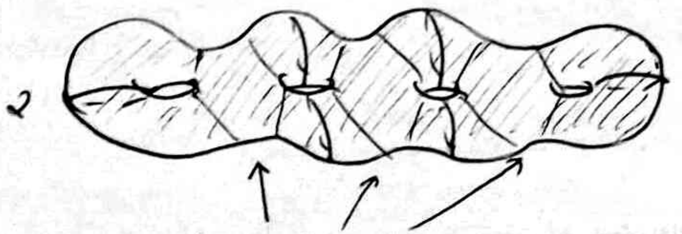
Recall. $\mu = \dfrac{A^2 - A^{-2}}{\sqrt{-2r}}$, A primitive 4r-th root of 1.

$$I_r(M) = \mu \cdot \langle \mu\omega_r, \dots, \mu\omega_r \rangle_D \langle \mu\omega_r \rangle_{U_-}^{\sigma}.$$

. three ways of getting a 3-mfd

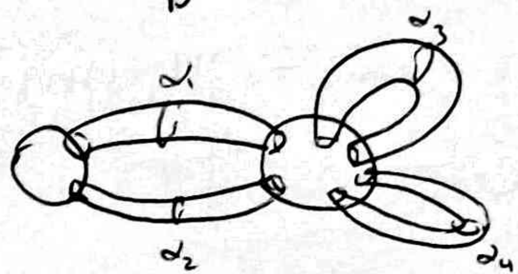
- Kirby diagram (Surgery) $\longrightarrow$ Reshetikhin-Turaev
- triangulation $\longrightarrow$ Turaev-Viro
- Heegaard Splitting $\longrightarrow$ Roberts invariant

Handle body.



$H_g = \{0\text{-handles}\} \cup \{1\text{-handles}\}$

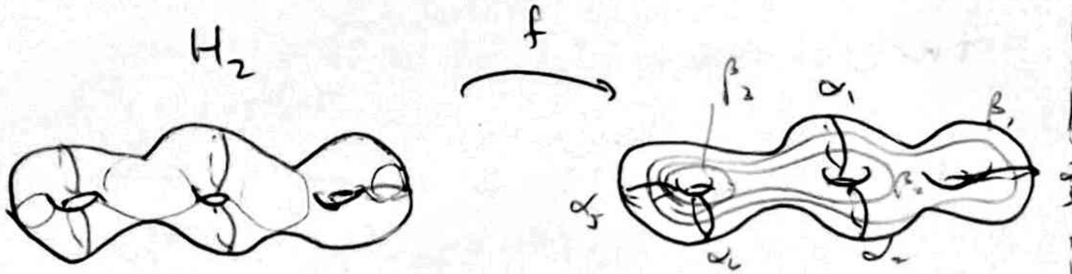
D^3 $D^2 \times D^1$.



$\alpha = \partial D^2 \times \{\frac{1}{2}\}$

Def. M closed oriented 3-mfd. A Heegaard Splitting of M consists of two handle bodies $H_1 \cong H_2 \cong H_g$ and a homeomorphism $f : \partial H_2 \longrightarrow \partial H_1$ s.t.

$$M \cong H_1 \cup_f H_2.$$



{2-handles, 3-handles} $\qquad$ {0-handles, 1-handles}

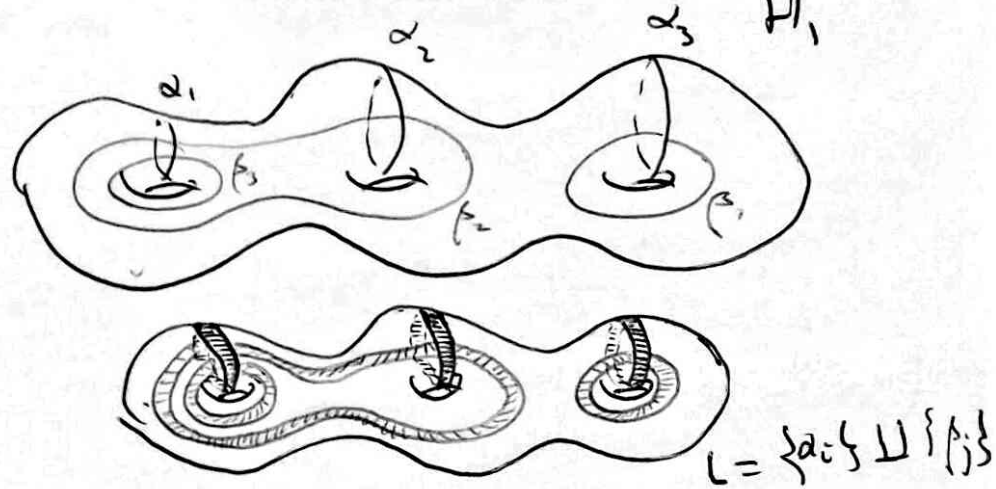
Heegaard splitting $\Longrightarrow$ handle decomposition

Notation:

d_i = # of i-handles $\quad \begin{cases} d_1 = \#\text{ of } \alpha\text{-curves} \\ d_2 = \#\text{ of } \beta\text{-curves} \end{cases}$

Thm. Every M has a Heegaard splitting.

Heegaard diagram:



Roberts' invariant (chain-mail) is constructed by the following steps.

1) embed H_1 into S^3
2) thicken the α- and β-curves along ∂H_1
3) push β-curves slightly into H_1 to get a framed link L in S^3.
4) Def:

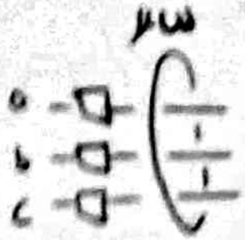
$$CH_r(M) = \mu^{d_0 + d_3} \langle \mu\omega_r, \dots, \mu\omega_r \rangle_L .$$

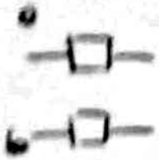
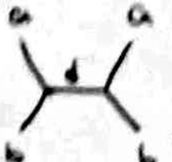
Thm1 (Roberts). $CH_r(M)$ defines an invariant of M, i.e., is independent of the Heegaard splitting of M and of embedding of H_1 into S^3.

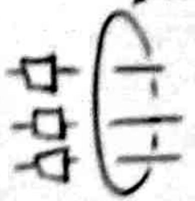
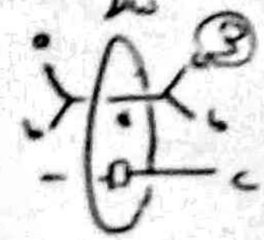
Thm2 (R) $\quad CH_r(M) = TV_r(M)$.

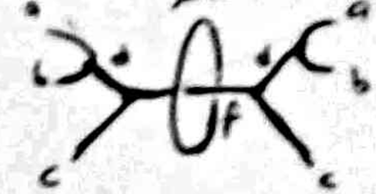
Thm3 (R) $\quad CH_r(M) = |I_r(M)|^2$

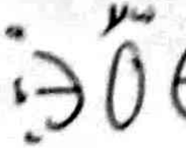
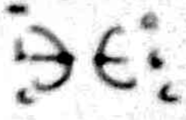
Fusion Rule

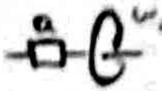
 $= \begin{cases} \dfrac{\mu^{-1}}{\theta(a,b,c)}$  $\quad \text{if } (a,b,c) \text{ adm.} \\ 0, \quad \text{if } (a,b,c) \text{ not adm for } a, b, c. \end{cases}$

Pf. Recall last time  $= \sum_d \dfrac{\theta_d}{\theta(a,b,d)}$ 

Then  $= \sum_d \dfrac{\theta_d}{\theta(a,b,d)}$ 

$= \sum_f \sum_d \dfrac{\theta_d \, \theta_f}{\theta(a,b,d)\,\theta(d,c,f)}$ 

$= \begin{cases} \dfrac{\theta_c}{\theta(a,b,c)\,\theta(a,b,c)}$  $= \dfrac{\mu^{-1}}{\theta(a,b,c)}$  $\quad \text{if } (a,b,c) \text{ adm.} \\ 0, \quad \text{if } (a,b,c) \text{ not adm, since } f \neq 0. \end{cases}$

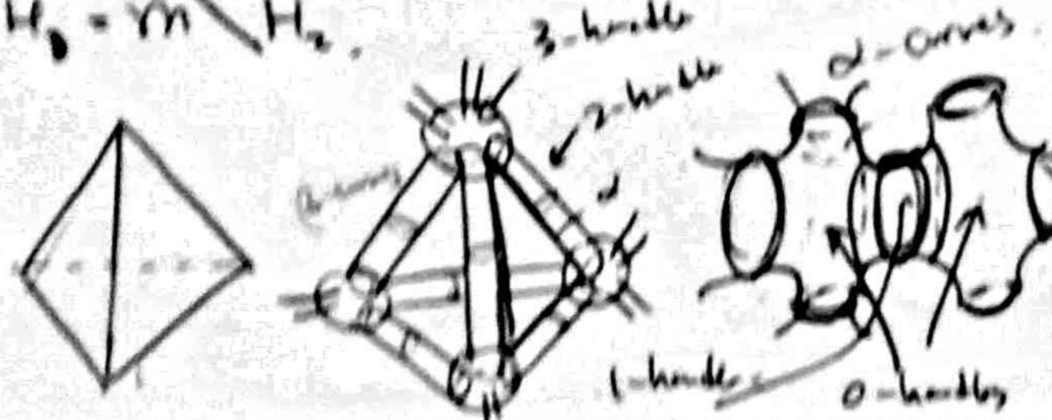
Recall  $= \begin{cases} 0, & a \neq 0 \\ O_{\omega_r}, & a = 0 \end{cases}$

~~Next theme:~~ Heegaard splitting from triangulation

$\mathcal{T}$. V, E, F, T sets of vertices, edges, faces, tetra.

H_1 = tubular nbhd of 1-skeleton $(E \cup V)$

$H_2 = M \setminus H_1$.



$\{0\text{-handles}\} \longleftrightarrow T \qquad d_0 = |T|$

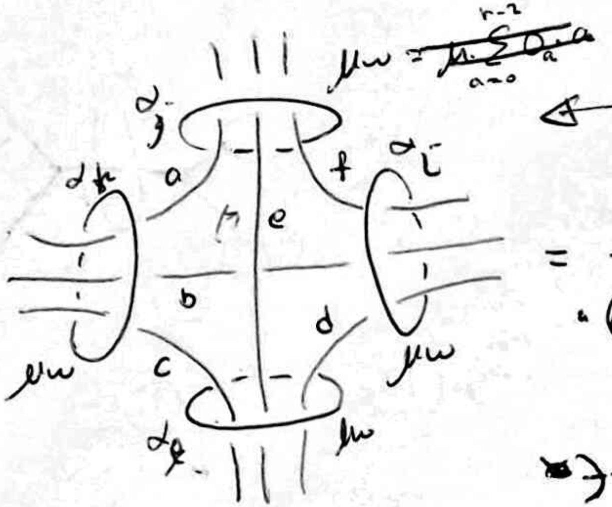
$\{1\text{-handles}\} \longleftrightarrow F \qquad d_1 = |F|$

$\{2\text{-handles}\} \longleftrightarrow E \qquad d_2 = |E|$

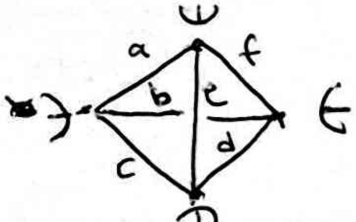
$\{3\text{-handles}\} \longleftrightarrow V \qquad d_3 = |V|$

• $L_{\mathcal{T}} = \{\alpha\} \cup \{\beta\}$

· Heegaard diagram has the property that every α-curve encloses 3 β-curves.



$$= \frac{\eta^{-4}}{\theta(a,b,c)\,\theta(a,e,f)\,\theta(c,d,e)\,\theta(b,d,f)}$$



pf of Thm 2. By Fusion Rule. $\mu\omega = \mu \sum_{a=0}^{r-2} O_a \cdot a$.

Then

$$CH_r(M) = \mu^{d_0+d_3} \langle \mu\omega, \ldots, \mu\omega \rangle_{L_T}$$

$$= \mu^{d_0+d_3+d_2-d_1} \sum_c \prod_e O \prod_f \Theta^{-1} \prod_\sigma \text{Tet}$$

$$= \eta^{-|V|} \sum_c \prod_e O \prod_f \Theta^{-1} \prod_\sigma \text{Tet}.$$

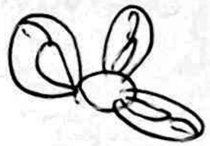
· $\chi(M) = d_0 - d_1 + d_2 - d_3 = 0$, $\Rightarrow d_2 - d_1 = d_3 - d_0$

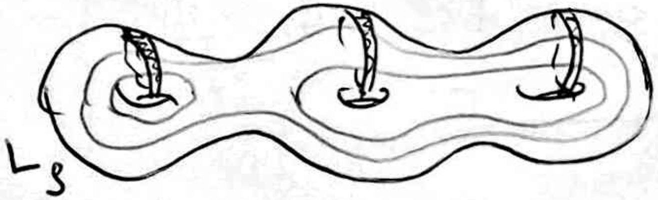
$\Rightarrow d_0 + d_3 + d_2 - d_1 = d_0 + d_3 + d_3 - d_0 = 2d_3 = 2|V|$.

Recall. $\eta = \frac{-2r}{(A^2 - A^{-2})^2} = \mu^{-2}$. □

· "Standard" Heegaard splitting.

(H_1, H_2, f) s.t. there are exactly one 0-handle and one 3-handle.





Then $CH_r(M) = \mu^2 \langle \mu\omega, \ldots, \mu\omega \rangle_{L_s}$.

Prop: $M_{L_s} = M \# (-M)$.

pf: By definition, $M_{L_s} = \partial X_{L_s}$, where X_{L_s} is 4-mfd from B^4 by attaching 2-handles along L_s.

~~Now let X'_{L_s} be from B^4 by attaching 1-handles along α-curves and 2-handles along β-curves~~

Now let X'_{L_s} be from B^4 by attaching 1-handles along α-curves and 2-handles along β-curves. (*)

Then

(i) $\partial X_{L_s} = \partial X'_{L_s}$.

(ii) $X'_{L_s} = m^{(2)} \times I \Rightarrow$ (iii) $\partial X'_{L_s} = m \# (-m)$.

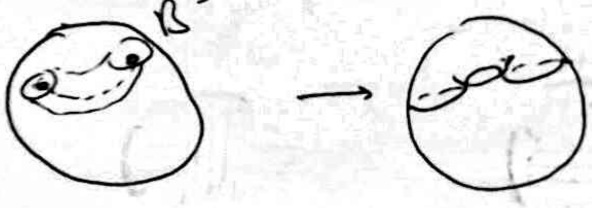
↑ 2-skeleton of m

(*). Let C be a circle in S^3 bounding a disk D_0 in $S^3 = \partial B^4$. Push D into B^4, and remove a tubular nbhd of D in B^4.

The resulting mfd is $M \cong B^3 \times S^1$, which is the same result as attaching a 1-handle to B^4. ~~The boundary of the result~~

$\partial M = S^2 \times S^1$

One way:



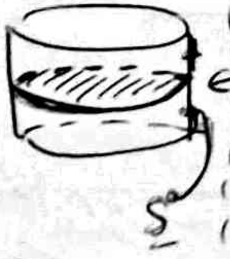
Another Way. $B^3 = D \times I$; $B^4 = D \times D$

$B^3 \setminus D$

$= D \times (I - \{\frac{1}{2}\})$

$= D \times (I \times S^0)$

$= B^3 \times S^0$



$B^4 \setminus D$

$= D \times (D - \{0\})$

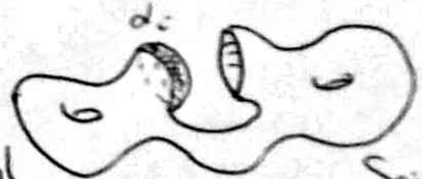
$= D \times (I \times S^1)$

$= B^3 \times S^1$

(i) Attaching 2-handle to $\bigcirc_0$ yields

$S^2 \times S^1$

why 0?



(ii) ~~$m^{(2)} = B^3 \cup \{1\text{-handles}\}$~~

$m^{(2)} = B^3 \cup \{1\text{-handles}\} \cup \{2\text{-handles}\}$

(iii)

$\partial(m^{(2)} \times I)$

$= m^{(2)} \cup (-m^{(2)})$

$= (m \setminus B^3) \cup$

Pf of Thm 3: $\left(I_r(M) = \eta \langle \mu\omega, \dots, \mu\omega \rangle_L \langle \mu\omega \rangle_{u_-}^{\sigma} \right)$

$CH(M) = \mu^2 \langle \mu\omega, \dots, \mu\omega \rangle_{L_S}$

$= \mu^2 \cdot \mu^{-1} \cdot I_r(M \# (-M)) \cdot \langle \mu\omega \rangle_{u_-}^{-\sigma(L_S)}$

Prop of II(2)

$= \mu^2 \cdot \mu^{-1} \cdot \mu^{-1} I_r(M) I_r(-M) \cdot \langle \mu\omega \rangle_{u_-}^{-\sigma(L_S)}$

Prop of I(1)

$= I_r(M) \overline{I_r(M)} \cdot \langle \mu\omega \rangle_{u_-}^{-\sigma(L_S)}$.

claim

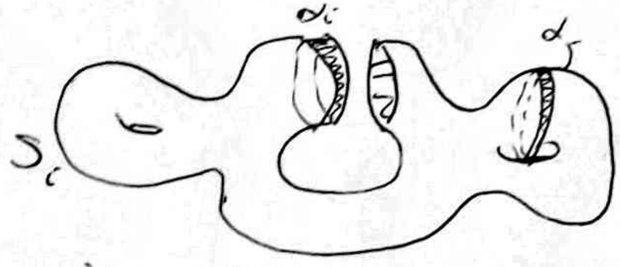
$= |I_r(M)|^2$.

Claim: $\sigma(L_S) = 0$.

Pf The linking matrix of L_S has the form

$$\begin{array}{cc} & \begin{array}{cc} \alpha & \beta \end{array} \\ \begin{array}{c} \alpha \\ \beta \end{array} & \left(\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array} \right) \end{array},$$

since $lk(\alpha_i, \alpha_j) = lk(\beta_k, \beta_l) = 0$



$i(\alpha_j, S) = 0$.

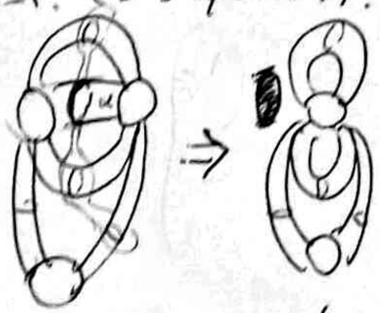
If $V = (V_1, V_2)$ is eigenvector of eigenvalue λ, then $V' = (-V_1, V_2)$ is e.v w/ e.v. $-\lambda$. □

Pf of Thm 1.

. Any two H.S. are differed by

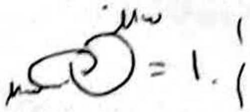
0-1, 1-2, 2-3 births or dies. 1- and 2-handle sliding.

0-1. (2-3 by duality).



band sum w/ other α_i's, until bounds a disk, then $\langle \mu\omega \rangle = \mu^{-1}$.

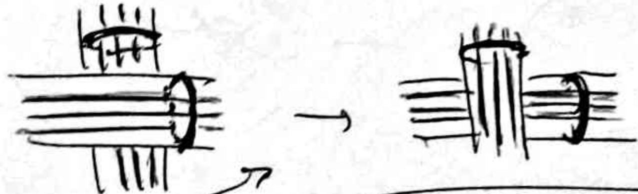
1-2



1-handle sliding doesn't chang the diagram.

2-handle sliding ✓

Any two embeddings of H_1 are differed by



band sum w/ α-circle.

Fenn-Rourk.