Math 251.504
Exam 1, version B
Solutions

1. (11 pts.) Find the equation of the plane containing the points \((1, 1, 1)\), \((-1, 1, 0)\), and \((3, 0, 1)\).
   Answer: To find the equation of a plane, we need a point in the plane and a vector normal to the plane. We’ve already got three points in the plane, so we need the normal. Call the given points \(P\), \(Q\), and \(R\). Then two vectors in the plane are \(\vec{PQ} = \langle -2, 0, -1 \rangle\) and \(\vec{PR} = \langle 2, -1, 0 \rangle\). Their cross product \(\vec{PQ} \times \vec{PR} = \langle -1, -2, 2 \rangle\) is perpendicular to both vectors in the plane, hence is normal to the plane. Plugging the first point in the formula for a plane, we get the equation is \(-x - 2y + 2z = 1\). (Using either of the other two points will give the same equation.

2. (6 points) For the surface \(x^2 - 9y^2 - 4z^2 = k\),

   (a) if \(k > 0\), what quadric surface is this? (i.e., name it) Answer: This is a hyperboloid of two sheets (Note: “hyperbola of two sheets” did not receive full credit. A hyperbola is a curve, not a surface.)

   (b) if \(k = 0\), what quadric surface is this? Answer: This is a cone (more specifically, an elliptic cone, but I accepted just “cone”).

   (c) if \(k < 0\), what quadric surface is this? Answer: This is a hyperboloid of one sheet.

3. (10 pts.) At what point(s) on the curve \(x = t^2, y = 2t, z = t^3 - 2\) is the tangent vector perpendicular to \((3, 0, 1)\)? Answer: Take \(\vec{r}(t) = \langle t^2, 2t, t^3 - 2 \rangle\) to be the position vector of the curve. Then \(\vec{r}'(t) = \langle 2t, 2, 3t^2 \rangle\) is always tangent to the curve. The \(t\) value for which this is orthogonal to \((3, 0, 1)\) satisfies \(\vec{r}'(t) \cdot (3, 0, 1) = 0\), i.e., \(6t + 3t^2 = 0\). This is solved for \(t = 0\) and \(t = -2\). Plugging in to find the points on the curve, we get \((0, 0, -2)\) and \((4, -4, -10)\).

4. (14 pts.) A particle is moving subject to a constant (vector) acceleration \(\vec{i} - 2\vec{j} + \vec{k}\), starting at the origin with initial velocity vector \(2\vec{i} + \vec{j} - \vec{k}\). Note: each part of this problem is worth 7 points, so if you only get seven points on a part, that’s probably why.

   (a) Find the position at any time \(t\). Answer: The velocity is an antiderivative of acceleration, so \(\vec{v}(t) = \langle t + C_1, -2t + C_2, t + C_3 \rangle\) for some constants \(C_1\), \(C_2\), \(C_3\). Since \(\vec{v}(0) = \langle 2, 1, -1 \rangle\), we get that velocity is \(\vec{v}(t) = \langle 2 + t, 1 - 2t, -1 + t \rangle\). Similarly, position is an antiderivative of velocity, and using the fact the \(\vec{r}(0) = \langle 0, 0, 0 \rangle\), we get that \(\vec{r}(t) = \langle \frac{1}{2}t^2 + 2t, -t^2 + t, \frac{1}{2}t^2 - t \rangle\) is the position at any time \(t\).

   (b) At what point other than the origin does it cross the plane \(z = 0\)? Answer: The \(t\) values for which the particle crosses the plane \(z = 0\) must satisfy \(\frac{1}{2}t^2 - t = 0\), so that \(t = 0\) or \(t = 2\). The value \(t = 0\) gives us the origin, so we want the other value: when \(t = 2\) the particle is at \((6, -2, 0)\), which is the other point where the particle crosses the plane \(z = 0\).

5. (12 pts.) Let \(f(x, y) = \sqrt{9 - x^2 - 9y^2}\).
(a) Sketch the domain of \( f \). Answer: The domain of \( f \) is the set of all points \( (x, y) \) which you can plug into \( f \). For the square root to be defined we need \( 9 - x^2 - 9y^2 \geq 0 \). This is an ellipse and its interior, shaded in green below:

(b) What is the range of \( f \)? Answer: The range is the set of all values which come out of \( f \). We can get values down to zero (never negative, since \( \sqrt{a} \) is defined to the the non-negative number whose square is \( a \)), but we can’t get any values above 3, since there’s no way that \( 9 - x^2 - 9y^2 \) can get above 9: we’re subtracting something non-negative from 9. Therefore the range is \([0, 3]\).

6. (15 pts.) Show that the set of all \((x, y, z)\) so that the distance from \((x, y, z)\) to \((3, -3, 0)\) equals twice the distance from \((x, y, z)\) to \((0, 0, 0)\) is a sphere, and find the center and radius of the sphere. Answer: Using the distance formula and translating the condition given to a formula, we get

\[
\sqrt{(x - 3)^2 + (y + 3)^2 + z^2} = 2\sqrt{x^2 + y^2 + z^2}.
\]

Square both sides:

\[
x^2 - 6x + 9 + y^2 + 6y + 9 + z^2 = 4x^2 + 4y^2 + 4z^2,
\]

which simplifies to

\[
3x^2 + 6x + 3y^2 - 6y + 3z^2 = 18.
\]

Divide by three and complete squares to get \((x + 1)^2 + (y - 1)^2 + z^2 = 8\), so this is a sphere with center \((-1, 1, 0)\) and radius \(\sqrt{8}\).

7. (10 pts.) An ant crawls along the curve \( x = \cos 3t, \ y = \sin 3t, \ z = t \), starting at \((1, 0, 0)\) in the direction of increasing \( z \). What is his location \((x, y, z)\) coordinates) after crawling 3 units? Answer: Take the position to be \( \vec{r}(t) = (\cos 3t, \sin 3t, t) \). The ant starts at \( t = 0 \) (which gives the point \((1, 0, 0)\)). The distance he’s crawled to time \( t \) is

\[
s = \int_0^t |r'(u)| \ du = \int_0^t \sqrt{9\sin^2 3u + 9\cos^2 3u + 1} \ du = \sqrt{10}t,
\]
since $\vec{r}'(t) = (-3 \sin 3t, 3 \cos 3t, 1)$. So, he’s crawled 3 units when $t = \frac{3}{\sqrt{10}}$, and his location at that time is $\left( \cos \left(2 \cdot \frac{3}{\sqrt{10}}\right), \sin \left(2 \cdot \frac{3}{\sqrt{10}}\right), \frac{3}{\sqrt{10}}\right)$.

8. (10 pts.) Does $\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + 2y^2}$ exist? If so, what is it? If not, why not? Support your answer. Answer: The limit does not exist. Let $g(x, y) = \frac{xy}{x^2 + 2y^2}$. Approaching the origin along the $x$ axis, we get

$$\lim_{x \to 0} g(x, 0) = \lim_{x \to 0} \frac{0}{x^2} = 0,$$

(It’s 0 for all $x \neq 0$, so the limit is zero.) Along the line $y = x$ we get

$$\lim_{x \to 0} g(x, x) = \lim_{x \to 0} \frac{x^2}{3x^2} = \frac{1}{3}.$$ 

Since we are getting different values by approaching the origin along different curves, the two dimensional limit does not exist.
9. (12 pts.) Match each of the following polar curves with the graphs below. *Parts d and e you should know. The rest you can do by doing a rough plot in Cartesian coordinates and converting it to polar coordinates.*

(a) $r = \theta$. *Answer: VI.*
(b) $r = \theta \cos \theta$. *Answer: IV.*
(c) $r = \sin (\theta/3)$. *Answer: I.*
(d) $r = \sin 3\theta$. *Answer: V.*
(e) $r = \cos 3\theta$. *Answer: III.*
(f) $r = 2 + \sin 3\theta$. *Answer: II.*