Conversion from spherical to cartesian coordinates:
\[ x = \rho \sin \phi \cos \theta \]
\[ y = \rho \sin \phi \sin \theta \]
\[ z = \rho \cos \phi \]
\[ dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \]

1. (12 pts.) A lamina is in the shape of the triangle with vertices \((0,0), (1,0)\) and \((1,2)\). Find \(\bar{y}\) of the center of mass, if density at any point is proportional to the distance to the \(y\) axis.

\textbf{Solution:} First off, notice that the distance of a point \((x,y)\) to the \(y\) axis is the \(x\) coordinate, so that \(\rho = kx\). Then,

\[ \bar{y} = \frac{\int_0^1 \int_0^{2x} kxy \, dy \, dx}{\int_0^1 \int_0^{2x} kx \, dy \, dx} = \frac{3}{4}. \]

2. (16 pts.) Suppose that \(E\) is the region in space bounded below by the paraboloid \(z = x^2 + 4y^2 - 3\) and above by the plane \(z = 1\). Set up (but do not evaluate) \(\iiint_E z \, dV\) as an iterated integral
   
   \textbf{(a) in the order } dz \, dy \, dx. \textbf{ Solution:} The \(z\) limits are pretty clear. To get the \(x\) and \(y\) limits, you have to project the intersection of the paraboloid and the plane \(z = 1\) into the \(xy\) plane, to get the ellipse \(x^2 + 4y^2 = 4\). Therefore the integral is
   \[ \int_{-2}^{2} \int_{-\sqrt{1 - \frac{x^2}{4}}}^{\sqrt{1 - \frac{x^2}{4}}} \int_{\frac{z^2}{x^2 + 4y^2 - 3}}^{1} z \, dz \, dy \, dx. \]

   \textbf{(b) in the order } dx \, dy \, dz. \textbf{ Solution:} For the inner limits, you go from one part of the paraboloid to the other. The projection into the \(y\), \(z\) plane is bounded by the parabola \(z = 4y^2 - 3\) and the line \(z = 1\). So, in the order \(dx \, dy \, dz\) the integral is
   \[ \int_{-3}^{1} \int_{-\sqrt{z + 3 - 4y^2}}^{\sqrt{z + 3 - 4y^2}} \int_{\frac{z^2}{x^2 + 4y^2 - 3}}^{1} z \, dx \, dy \, dz. \]

3. (10 pts.) Use Green’s theorem to evaluate \(\oint_C \left( e^{x^2} + 2y^3 \right) \, dx + \left( e^{y^2} - 2x^3 \right) \, dy\), where \(C\) is the upper half of the circle \(x^2 + y^2 = R^2\), followed by the line segment from \((-R,0)\) to \((R,0)\), oriented positively. \textbf{Solution:} By Green’s theorem, the line integral must equal
\[ \iint_D \frac{\partial}{\partial x} \left( e^{y^2} - 2x^3 \right) - \frac{\partial}{\partial y} \left( e^{x^2} + 2y^3 \right) \, dA = \iint_D (-6x^2 - 6y^2) \, dA, \]
where \(D\) is the upper half disk of radius \(R\). This is best done in polar:
\[ \int_0^\pi \int_0^R (-6r^2 \, dr \, d\theta = -3 \pi R^4. \]
4. (16 pts.) Convert \( \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} x^2 \, dz \, dy \, dx \) to an integral in:

(a) cylindrical coordinates. *Solution:* That \( z \) goes from \( z = r \) to \( z = 1 \) is pretty clear. The \( x \) and \( y \) limits correspond to a quarter of the unit circle, so in cylindrical coordinates the integral is
\[
\int_0^{\pi/2} \int_0^1 \int_r^1 r^4 \cos^2 \theta \sin \theta \, dz \, dr \, d\theta,
\]
not forgetting that \( dV = r \, dz \, dr \, d\theta \).

(b) spherical coordinates. *Solution:* The plane \( z = 1 \) is \( \rho \cos \phi = 1 \), so \( \rho \) goes from the origin out to \( \rho = \sec \phi \). Since we’re in a quarter of a cone, the integral is
\[
\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sec \phi} \left( \rho \sin \phi \cos \theta \right)^2 \left( \rho \sin \phi \sin \theta \right) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^5 \sin^4 \phi \cos^2 \theta \sin \theta \, d\rho \, d\phi \, d\theta.
\]
You can do the \( d\phi \) and \( d\theta \) integrals in either order, but the \( d\rho \) integral really better be the inner one.

5. (12 pts.) Evaluate \( \int_C (2xy + \sin x) \, dx + (x^2 + 2e^y) \, dy \), where \( C \) is an arbitrary smooth curve starting at \((2,1)\) and ending at \((0,0)\). *Solution:* Since we aren’t given the curve, for the problem to make sense we’ve got to integrating \( \int (\nabla f) \cdot d\vec{r} \) for some function, and then we can use the fundamental theorem of line integrals. Just to be sure, though, we check that
\[
\frac{\partial}{\partial y} (2xy + \sin x) = \frac{\partial}{\partial x} (x^2 + 2e^y) = 2x.
\]
So it is a gradient, and we need to recover \( f \). Since \( f_x = 2xy+\sin x \), \( f(x,y) = x^2y-\cos x + g(y) \) for some function \( g \). Differentiating this with respect to \( y \), we get that \( f_y = x^2 + g'(y) \), and, comparing with what \( f_y \) is supposed to be, we see that \( g'(y) = 2e^y \), therefore \( g(y) = 2e^y + C \). Since \( C \) will cancel anyway, we drop it, to get \( f(x,y) = x^2y - \cos x + 2e^y \). Then the line integral must equal
\[
f(0,0) - f(2,1) = -1 + 2 - (4 - \cos 2 + 2e) = -3 + \cos 2 - 2e.
\]

6. (10 pts.) Convert \( \int_0^2 \int_0^{\sqrt[3]{x^2+y^2+1}} dy \, dx \) to an integral in polar coordinates (but do not evaluate it). *Solution:* \( r \) goes from the line \( x = 2 \) to the origin. But \( x = 2 \) is \( r \cos \theta = 2 \) or \( r = 2 \sec \theta \), so the integral is
\[
\int_0^{\pi/3} \int_0^{2 \sec \theta} \frac{r \, dr \, d\theta}{r^2 + 1},
\]
since the line \( y = x \sqrt{3} \) is given by \( \theta = \frac{\pi}{3} \).
7. (12 pts.) Evaluate the line integral \( \int_C x\,ds \), where \( C \) is the portion of the parabola \( y = x^2 \) from \((0, 0)\) to \((1, 1)\). Solution: Parameterize the parabola by \( x = t, \ y = t^2, \ 0 \leq t \leq 1 \). Then 
\[
 ds = \sqrt{(x')^2 + (y')^2} \, dt = \sqrt{1 + 4t^2} \, dt, 
\]
and the integral becomes 
\[
 \int_0^1 t \sqrt{1 + 4t^2} \, dt = \left. \frac{1}{12} (1 + 4t^2)^{3/2} \right|_0^1 = \frac{1}{12} (5^{3/2} - 1), 
\]
where I used the substitution \( u = 1 + 4t^2 \).

8. (12 pts.) Match each of the following vector fields with the plots below

(a) \( \langle \cos(5x), \sin(5y) \rangle \). Answer: III.
(b) \( \langle -x + y, x + 2y \rangle \). Answer: I.
(c) \( \langle \cos(5y), \sin(5x) \rangle \). Answer: V.
(d) \( \langle x^2, y^2 \rangle \). Answer: II.
(e) \( \langle y^2, x^2 \rangle \). Answer: IV.
(f) \( \langle x + y, -x + y \rangle \). Answer: VI.