1. (20 pts.) Determine the interval of convergence of the power series

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}8^n} (x - 3)^{3n}. \]

**Solution:** It’s probably easiest to use the ratio test:

\[
\lim_{n \to \infty} \left| \frac{(-1)^{n+1}(x - 3)^{3(n+1)}}{(x - 3)^{3n}} / \left( \frac{\sqrt{n + 1}}{\sqrt{n}} \right) \right| = \lim_{n \to \infty} \frac{|x - 3|^3}{8} \sqrt{\frac{n}{n + 1}} = \frac{|x - 3|^3}{8}.
\]

If this limit is less than 1, the series converges absolutely, and if the limit is larger than 1, the series diverges since its terms don’t go to zero. Thus we can say that \(|x - 3| < 2\) implies convergence and \(|x - 3| > 2\) implies divergence. So, we certainly have convergence on \(1 < x < 5\), and all that’s left is to check the endpoints. At \(x = 5\), the series is

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}8^n} (2)^{3n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}, \]

which converges by the alternating series test. At \(x = 1\) the series is

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}8^n} (-2)^{3n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^{3n}}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}, \]

a divergent \(p\)-series. Therefore the interval of convergence is \((1, 5]\).

2. (20 pts.) Prove that \( \lim_{n \to \infty} \left( \frac{3}{n^2} \cdot \frac{1}{n^2} \right) = (0, 0) \) from the definition, i.e., find (with proof) an \(N(\varepsilon)\) so that \(n \geq N\) implies that

\[ \left\| \left( \frac{3}{n^2}, \frac{1}{n^2} \right) - (0, 0) \right\| < \varepsilon. \]
This is the standard Euclidean norm. Solution: First, to find $N$. We want $\| \left( \frac{3}{n}, \frac{1}{n^2} \right) - (0, 0) \| < \varepsilon$, i.e., $\sqrt{\frac{9}{n^2} + \frac{1}{n^4}} < \varepsilon$. Since $\frac{1}{n^4} < \frac{1}{n^2}$ for all $n > 1$, the inequality we want will hold if $\frac{\sqrt{9n^2}}{\varepsilon} \leq n$, as long as we also require $n > 1$. So, we’re coming up with $N(\varepsilon) = \max \left( 2, \frac{\sqrt{9}}{\varepsilon} \right)$. Here’s the proof that our $N(\varepsilon)$ works: Let $\varepsilon > 0$ be arbitrary. Take $n \geq \max \left( 2, \frac{\sqrt{9}}{\varepsilon} \right)$. Since $n \geq \sqrt{\frac{9}{\varepsilon}}$, $\sqrt{\frac{9}{n^2} + \frac{1}{n^4}} \leq \varepsilon$, and since $\frac{1}{n^4} < \frac{1}{n^2}$ for $n > 1$, it follows that

$$\sqrt{\frac{9}{n^2} + \frac{1}{n^2}} \leq \varepsilon.$$

3. (20 pts.) Suppose that $x, y \in \mathbb{R}^n$, and that $\| x + y \| < 3$ and $\| x - 2y \| < 5$. Find, with proof, upper bounds for $\| x \|$ and $\| y \|$. (Standard Euclidean norm) Solution: Since $3x = 2(x + y) + (x - 2y)$, by the triangle inequality we have

$$\| x \| = \left\| \frac{2}{3} (x + y) + \frac{1}{3} (x - 2y) \right\| \leq \frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 5 = \frac{11}{3}.$$

and since $3y = (x + y) - (x - 2y)$, we have

$$\| y \| = \left\| \frac{1}{3} (x + y) + \frac{1}{3} (-1) (x - 2y) \right\| \leq \frac{1}{3} \| x + y \| + \frac{1}{3} \| x - 2y \| < \frac{3}{3} + \frac{5}{3} = \frac{8}{3}.$$

4. (20 pts.) Define the term in italics:

(a) $\Omega \subseteq \mathbb{R}^n$ is an open set. Solution: $\Omega$ is open iff for every $x \in \Omega$ there exists $\varepsilon > 0$ so that $B_{\varepsilon}(x) \subseteq \Omega$.

(b) the boundary of a set $A \subseteq \mathbb{R}^n$. Solution: The boundary of $A$ consists of all points $x \in \mathbb{R}^n$ so that for every $r > 0$, $B_r(x)$ contains at least one point of $A$ and at least one point of $A^c$ (the complement of $A$). (Note: the fact that $\partial A = \overline{A} - A^c$ is true, but it’s not the definition of boundary.)
(c) a function $f(x)$ is analytic on an interval $(a, b)$. Solution: $f$ is analytic on $(a, b)$ iff for every $x_0 \in (a, b)$ there exists a power series $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ which equals $f(x)$ on some open interval containing $x_0$.

(d) the sup norm (or $\|x\|_{\infty}$) for a vector $x \in \mathbb{R}^n$. Solution: for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\|x\|_{\infty} = \max(|x_1|, \cdots, |x_n|)$.

5. (20 pts.) Suppose that $A$ and $B$ are open sets in $\mathbb{R}^n$, and that $A \cap B = \emptyset$. Prove that $A \cap \overline{B} = \emptyset$, where $\overline{B}$ is the closure of $B$. Solution: Since $A \cap B = \emptyset$, $B \subseteq A^c$. But $A^c$ is closed (since $A$ is open), and $\overline{B}$ is the smallest closed set containing $B$, thus $\overline{B} \subseteq A^c$. Since $\overline{B} \subseteq A^c$, it follows that $A \cap \overline{B} = \emptyset$, as desired.