Math 410.500: answers to exam 2

1. (a) \( U \subseteq \mathbb{R}^n \) is open if and only if for every \( a \in U \) there exists \( r > 0 \) so that \( B_r(a) \subseteq U \).

(b) The boundary of \( U \) is the set of all points \( x \in \mathbb{R}^n \) so that \( \forall r > 0 \), both \( B_r(x) \cap U \neq \emptyset \) and \( B_r(x) \cap U^c \neq \emptyset \).

(c) \( E \subseteq \mathbb{R}^n \) is connected if there does not exist a pair of open sets \( A, B \) with \( E \subseteq A \cup B, E \cap A \neq \emptyset, E \cap B \neq \emptyset \), and \( A \cap B = \emptyset \) (my definition from class). \( \text{or: } E \subseteq \mathbb{R}^n \) is connected if there does not exist a pair of subsets \( U \) and \( V \) of \( E \), \( U \) and \( V \) both relatively open in \( E \), \( E = U \cup V, U \neq \emptyset, V \neq \emptyset \), and \( U \cap V = \emptyset \) (book's definition).

2. Since \( C \) is relatively closed in \( E \), then by definition there exists a closed set \( K \) so that \( C = K \cap E \). Since the \( x_k \)'s are in \( C \), they are in \( K \). Since the \( x_k \)'s converge and \( K \) is closed, their limit must be in \( K \), i.e., \( x \in K \). But we're already given that \( x \in E \), thus \( x \in E \cap K = C \).

3. (a) False: a counterexample is \( C_k = \left[ \frac{1}{k}, 2 \right] \). Then \( \bigcup_{k=1}^{\infty} C_k = (0, 2] \), which is not closed.

(b) True: Theorem 8.24 (iii).

(c) True: By the Heine-Borel Theorem, if \( H \) is closed and bounded, \( H \) must be compact, which means that every open cover of \( H \) has a finite subcover.

(d) False: a counterexample is \( x_k = (k, 0, \ldots, 0) \). If a subsequence \( \{x_{k_j}\} \) converged, we'd have to have \( \|x_{k_j}\| \) bounded. But \( \|x_{k_j}\| = k_j \to \infty \).

(e) True: Theorem 9.30.

(f) False: Example 9.28 gives a counterexample.

4. (a) For all \( x, y \in \mathbb{R}^n \), \( \|x \cdot y\| \leq \|x\| \|y\| \). (I took off a point if you didn't quantify \( x \) and \( y \).)

(b) Of course, part a was a hint. Pick \( \varepsilon > 0 \). Case 1: if \( a = 0 \) (the zero vector) then \( f(x) \) is identically 0, and uniform continuity follows quickly: set \( \delta \) to be 1. If \( \|x_1 - x_2\| < 1 \), then \( |f(x_1) - f(x_2)| = 0 < \varepsilon \). Case 2: if \( a \neq 0 \), set \( \delta \) to be \( \frac{\varepsilon}{\|a\|} \). Then if \( \|x_1 - x_2\| < \frac{\varepsilon}{\|a\|} \), we have

\[
|f(x_1) - f(x_2)| = |a \cdot x_1 - a \cdot x_2| \\
= |a \cdot (x_1 - x_2)| \\
\leq \|a\| \|x_1 - x_2\|, \text{ (by Cauchy-Schwartz)} \\
< \|a\| \frac{\varepsilon}{\|a\|} = \varepsilon.
\]

Since in either case, \( \delta \) depends only on \( \varepsilon \), \( f \) is uniformly continuous.
5. The first coordinate is alternating, the second coordinate repeats in blocks of 3, so the points in $x_k$ repeat in blocks of 6. So, it makes sense to look at every sixth term:

$$x_{6j} = \left( (-1)^{6j} , \sin \left( \frac{12j}{3} \pi \right) \right) = (1, 0),$$

so $\{x_{6j}\}$ converges to $(1, 0)$. Now look at other terms 6 apart, say $\{x_{6j+3}\}$:

$$x_{6j+3} = \left( (-1)^{6j+3} , \sin \left( \frac{12j}{3} + 2 \right) \pi \right) = (-1, 0),$$

so this subsequence converges to $(-1, 0)$.

6. Away from the origin, we may use the quotient rule to say that $f_x = \frac{x^4 + 3x^2y^2}{(x^2 + y^2)^2}$ (after a little simplification). At the origin, we must use the definition of the partial:

$$f_x (0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h^3}{h} = \lim_{h \to 0} h = 1.$$

Thus

$$f_x (x, y) = \begin{cases} x^4 + 3x^2y^2 & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0). \end{cases}$$

Now we must decide where $f_x$ is continuous. Away from the origin, $f_x$ is a rational function which is automatically continuous on its domain (i.e., where you aren’t dividing by 0). So, the only remaining question is whether this is continuous at the origin. But along the $y$ axis, $x$ is 0, so $f_x (0, y) = 0$ for $y \neq 0$, and it is impossible to have $\lim_{(x,y) \to (0,0)} f_x (x, y) = 1 = f_x (0, 0)$, and $f_x$ is not continuous at $(0, 0)$ (In fact, the limit of $f_x$ doesn’t exist at all at $(0, 0)$, since $f_x(x, 0) = 1$ for $x \neq 0$.)