Math 410.500
Exam 2
3/30/05

There are problems on both sides of this sheet!

1. (18 pts.) Define the term in italics:
   (a) $U \subseteq \mathbb{R}^n$ is an open set.
   (b) What is the boundary of a set $U \subseteq \mathbb{R}^n$?
   (c) $E \subseteq \mathbb{R}^n$ is a connected set. (Either the definition from the book or the definition I gave in class is acceptable.)

2. (18 pts.) Prove that if $C \subseteq E$ is relatively closed in $E$ and if $\{x_k\}$ is a sequence of points of $C$ which converge to a point $x \in E$, then in fact $x$ must be an element of $C$. (This is part of a problem from homework, so of course you can’t just quote that problem.)

3. True or false? (3 pts. each) (Your answer should just be “T” or “F”: I’m not looking for explanations or examples.)
   (a) If $C_\alpha$ is closed for all $\alpha \in A$, where $A$ is some indexing set which might be infinite, then $\bigcup_{\alpha \in A} C_\alpha$ is closed.
   (b) If $C_\alpha$ is closed for all $\alpha \in A$, where $A$ is some indexing set which might be infinite, then $\bigcap_{\alpha \in A} C_\alpha$ is closed.
   (c) If $H \subseteq \mathbb{R}^n$ is a closed and bounded set, then any open cover of $H$ must have a finite sub-cover.
   (d) Any sequence of points in $\mathbb{R}^n$ must have a convergent subsequence.
   (e) If $f : E \rightarrow \mathbb{R}^m$ is continuous and $E$ is connected, then $f(E)$ must be connected.
   (f) If $f : E \rightarrow \mathbb{R}^m$ is continuous and $E$ is open, then $f(E)$ must be open.

4. 
   (a) (6 pts.) What is the Cauchy-Schwartz inequality in $\mathbb{R}^n$?
   (b) (10 pts.) Suppose that $a \in \mathbb{R}^n$ is a fixed element of $\mathbb{R}^n$. Define $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by $f(x) = a \cdot x$, where this is the dot product. Prove that $f$ is uniformly continuous on $\mathbb{R}^n$. 

1
5. (12 pts.) Let \( x_k = \left( (-1)^k, \sin \left( \frac{2k\pi}{3} \right) \right) \). Find two different convergent subsequences which converge to different limits. (Specify the subsequences by replacing \( k \) by some function of \( j \), e.g., \( x_{5j+3} \), and be sure to give the limit for each subsequence.)

6. (18 pts.) For

\[
   f(x, y) = \begin{cases} 
   \frac{x^3}{x^2 + y^2} & (x, y) \neq (0, 0), \\
   0 & (x, y) = (0, 0),
   \end{cases}
\]

determine \( f_x (x, y) \) (the partial derivative with respect to \( x \)), and determine where \( f_x (x, y) \) is continuous.