Math 410.500
Exam 3
4/29/05

There are problems on both sides of this sheet!

1. (10 pts.) True or false? (no explanation is needed)
   (a) If $f$ is differentiable at $a \in \mathbb{R}^n$ then $f$ is continuous at $a$.
   (b) If all first order partials of $f$ exist at $a$ then $f$ is differentiable at $a$.
   (c) If all first order partials of $f$ exist at $a$ then $f$ is continuous at $a$.
   (d) If all first order partials of $f$ are continuous in an open set containing $a$ then $f$ is differentiable at $a$.
   (e) If $f$ is differentiable at $a$ then all first order partials of $f$ are continuous at $a$.

2. Define:
   (a) (5 pts.) The norm of a linear operator $T$. (We saw a couple of definitions: either one is fine.)
   (b) (5 pts.) A convex set.

3. Suppose that $f(a, b, c, d)$ and $g(a, b, c, d)$ are continuously differentiable on $\mathbb{R}^4$ and that $(a_0, b_0, c_0, d_0)$ solves the system

   $f(a, b, c, d) = 0$
   $g(a, b, c, d) = 0$.

   (a) (10 pts.) What additional condition do we need to conclude by the Implicit Function Theorem that the above system defines $a$ and $b$ implicitly as differentiable functions of $c$ and $d$ in an open set containing $(c_0, d_0)$?
   (b) (10 pts.) If that condition is met, find $\frac{\partial a}{\partial c}(c_0, d_0)$ in terms of partial derivatives of $f$ and $g$. 

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4. For \( f(x, y) = \frac{1}{x + 2y} \) at \( a = (1, 1) \),

(a) (5 pts.) write Taylor’s formula with \( p = 1 \).

(b) (15 pts.) write Taylor’s formula with \( p = 2 \).

5. Define \( f(x, y) \) by

\[
 f(x, y) = \begin{cases} 
 \frac{x|x|}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0); \\
 0, & \text{if } (x, y) = (0, 0). 
\end{cases}
\]

(a) (5 pts.) Find \( f_x(0, 0) \) and \( f_y(0, 0) \). (Notice that I’m not asking for these partials at any other point!)

(b) (15 pts.) Prove that \( f \) is not differentiable at \((0, 0)\).

6. (20 pts.) Determine

\[
 \lim_{y \to 0} \frac{\int_1^2 (x^3 + y)^{1/3} \, dx - \frac{3}{2}}{y},
\]

verifying the hypotheses of the theorems that you’re using.