as we approach the origin. Thus
\[ \lim_{(h,k) \to (0,0)} \frac{f(1+h,2+k) - [f(1,2) + 2h + 8k]}{\sqrt{h^2 + k^2}} = 0, \]
and \( f \) is differentiable at \((1,2)\).

I’ll continue looking at functions from \( \mathbb{R}^2 \) to \( \mathbb{R} \) for a while before generalizing.

**Proposition:** If \( f(x,y) \) is differentiable at \((a,b)\), then the first order partials of \( f \) exist at \((a,b)\), and the \( A \) and \( B \) from the definition of differentiability satisfy \( A = f_x(a,b) \) and \( B = f_y(a,b) \).

**Proof:** We are given that
\[ \lim_{(h,k) \to (0,0)} \frac{f(a + h, b + k) - [f(a, b) + Ah + Bk]}{\| (h, k) \|} = 0. \]
Approaching the origin in the \( h, k \) plane along the line \( k = 0 \), we still have to get zero. Thus
\[ \lim_{h \to 0} \frac{f(a + h, b) - [f(a, b) + Ah]}{\sqrt{h^2}} = 0, \]
so that
\[ \lim_{h \to 0} \frac{f(a + h, b) - [f(a, b) + Ah]}{|h|} = 0. \]
But by looking at the one sided limits at \( h = 0 \) separately, it follows that
\[ \lim_{h \to 0} \frac{f(a + h, b) - [f(a, b) + Ah]}{h} = 0, \]
i.e.,
\( f_x(a,b) = A \). Similarly, \( f_y(a,b) = B \).

**Proposition:** If \( f(x,y) \) is differentiable at \((a,b)\), then \( f(x,y) \) is continuous at \((a,b)\).

**Proof:** To show continuity at \((a,b)\), it suffices to show that
\[ \lim_{(h,k) \to (0,0)} f(a + h, b + k) = f(a, b). \]
But
\[ (f(a + h, b + k) - f(a, b)) = \lim_{(h,k) \to (0,0)} (f(a + h, b + k) - [f(a, b) + Ah - Bk]) \]
\[ = \lim_{(h,k) \to (0,0)} \frac{(f(a + h, b + k) - f(a, b) - Ah - Bk)}{\sqrt{h^2 + k^2}} \sqrt{h^2 + k^2} \]
\[ = 0 \cdot 0 = 0. \]

**Example:** Let \( f(x,y) = (x^3 + y^3)^{1/3} \).

(1) What are \( f_x(0,0) \) and \( f_y(0,0) \)?

(2) Is \( f \) differentiable at \((0,0)\)?

©2012, T. Vogel
Solution:

(1) By definition,

\[ f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} \]

\[ = \lim_{h \to 0} \frac{h^3}{1/3} = 1, \]

and similarly \( f_y(0, 0) = 1 \). (Note that I didn’t compute the partials away from the origin, since that wasn’t asked for and they aren’t relevant.)

(2) If \( f \) is differentiable, then both \( A \) and \( B \) from the definition must equal 1, since the values of the partials are equal to 1. Thus to see if \( f \) is differentiable at \((0, 0)\), we must look at

\[ \lim_{(h, k) \to (0, 0)} \frac{(h^3 + k^3)^{1/3} - h - k}{\sqrt{h^2 + k^2}}. \]

Put this into polar coordinates, with \( h = r \cos \theta \) and \( k = r \sin \theta \). Factors of \( r \) will cancel, and the fraction turns into \((\sin^3 \theta + \cos^3 \theta)^{1/3} - \sin \theta - \cos \theta\). This is different along different \( \theta \) values (compare \( \theta = 0 \) and \( \theta = \frac{\pi}{4} \)), and since there are no \( r \)'s, as we approach the origin along different lines we get different limiting values. Thus the two dimensional limit as \((h, k)\) tends to \((0, 0)\) does not exist, and the function is not differentiable at the origin.

**TEST 3 UP TO HERE.**

Now we want to generalize this to \( f : \mathbb{R}^n \to \mathbb{R}^m \). The idea is this: \( f \) will be differentiable at \( a \) if \( f(a + h) - f(a) \) is pretty close to a linear transformation acting on \( h \). More formally,

**Definition:** Suppose that \( U \subseteq \mathbb{R}^n \), \( f : U \to \mathbb{R}^m \). Then \( f \) is said to be differentiable at \( a \in U \) if there exists an open ball centered at \( a \) which is contained in \( U \), and a linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) so that

\[ \lim_{h \to 0} \frac{f(a + h) - [f(a) + T(h)]}{\|h\|} = 0, \]

where this is a limit of vectors in \( \mathbb{R}^m \). Equivalently, we can write

\[ \lim_{h \to 0} \frac{\|f(a + h) - [f(a) + T(h)]\|}{\|h\|} = 0, \]

where this a limit in \( \mathbb{R} \) (of course, \( h \) is a vector in \( \mathbb{R}^n \)). I would call the linear transformation \( T \) the total derivative of \( f \) at \( a \). Our book would call the matrix of \( T \) the total derivative. Compare this with
the definition for $f : \mathbb{R}^2 \to \mathbb{R}$: the vector $h$ is $(h, k)$, and the linear transformation $T$ is

$$T \left( \begin{array}{c} h \\ k \end{array} \right) = \left( \begin{array}{cc} A & B \end{array} \right) \left( \begin{array}{c} h \\ k \end{array} \right).$$

**Example:** Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} xy \\ 2x^2 + y^2 \end{array} \right).$$

Verify that $f$ is differentiable at $\left( \begin{array}{c} 1 \\ 2 \end{array} \right)$ and find the matrix of its total derivative there.

**Solution:** We essentially want

$$f \left( \begin{array}{c} 1 + h_1 \\ 2 + h_2 \end{array} \right) - f \left( \begin{array}{c} 1 \\ 2 \end{array} \right)$$

to be a linear transformation acting on $h = \left( \begin{array}{c} h_1 \\ h_2 \end{array} \right)$ plus terms which are much smaller than $\|h\|$. So,

$$f \left( \begin{array}{c} 1 + h_1 \\ 2 + h_2 \end{array} \right) - f \left( \begin{array}{c} 1 \\ 2 \end{array} \right) = \left( \begin{array}{c} (1 + h_1)(2 + h_2) \\ 2(1 + h_1)^2 + (2 + h_2)^2 \end{array} \right) - \left( \begin{array}{c} 2 \\ 6 \end{array} \right)$$

$$= \left( \begin{array}{c} 2h_1 + h_2 + h_1h_2 \\ 4h_1 + 4h_2 + 2h_1^2 + h_2^2 \end{array} \right)$$

$$= \left( \begin{array}{cc} 2 & 1 \\ 4 & 4 \end{array} \right) \left( \begin{array}{c} h_1 \\ h_2 \end{array} \right) + \left( \begin{array}{c} h_1h_2 \\ 2h_1^2 + h_2^2 \end{array} \right).$$

If we show that

$$\lim_{h \to 0} \frac{h_1h_2}{\sqrt{h_1^2 + h_2^2}} \leq \left( \begin{array}{c} 0 \\ 0 \end{array} \right),$$

we’re done, and we get to say that the matrix of the total derivative is $\left( \begin{array}{cc} 2 & 1 \\ 4 & 4 \end{array} \right)$. To see this, look at each component and put them into polar coordinates. In polar coordinates (letting $h_1 = r \cos \theta$ and $h_2 = r \sin \theta$), we have

$$\left| \frac{h_1h_2}{\sqrt{h_1^2 + h_2^2}} \right| = |r \sin \theta \cos \theta| \leq r,$$

so

$$-\sqrt{h_1^2 + h_2^2} \leq \frac{h_1h_2}{\sqrt{h_1^2 + h_2^2}} \leq \sqrt{h_1^2 + h_2^2}.$$
so \( \frac{h_1 h_2}{\sqrt{h_1^2 + h_2^2}} \rightarrow 0 \) as \( h \rightarrow 0 \). Similarly,

\[
0 \leq \frac{2h_1^2 + h_2^2}{\sqrt{h_1^2 + h_2^2}} = \frac{2r^2 \cos^2 \theta + r^2 \sin^2 \theta}{r} \leq 3r
\]

\[
= 3\sqrt{h_1^2 + h_2^2} \rightarrow 0.
\]

The first hint that the definition of differentiability given above might be useful is the following result.

**Theorem:** If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \).

**Proof:** First, notice that for any linear transformation \( T \) we must have \( \lim_{h \rightarrow 0} T(h) = 0 \). The reason is the operator norm inequality from 8.2. Take \( \varepsilon > 0 \). Since \( \|T(h)\| \leq \|T\| \|h\| \), we have \( \|T(h) - 0\| < \varepsilon \) if \( \|h\| < \frac{\varepsilon}{\|T\|} \). (The special case \( \|T\| = 0 \) is no problem.) So, suppose that \( f \) is differentiable at \( a \) with total derivative \( T \). Then

\[
\lim_{h \rightarrow 0} (f(a + h) - f(a)) = \lim_{h \rightarrow 0} (f(a + h) - f(a) - T(h))
\]

\[
= \lim_{h \rightarrow 0} \left( \frac{f(a + h) - f(a) - T(h)}{\|h\|} \right) \|h\|
\]

\[
= \lim_{h \rightarrow 0} \left( \frac{f(a + h) - f(a) - T(h)}{\|h\|} \right) \lim_{h \rightarrow 0} \|h\| = 0,
\]

where the justification of the third step is that, by assumption,

\[
\lim_{h \rightarrow 0} \left( \frac{f(a + h) - f(a) - T(h)}{\|h\|} \right) = 0
\]

exists and is 0.

**Questions:** How do we find the matrix of the total derivative? Are there reasonable criteria for differentiability?

First question first: Suppose that

\[
f = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{pmatrix},
\]

and that all \( f_i \) have partials of first order existing at a point \( a \).

**Definition:** The Jacobian matrix of \( f \) at \( a \) is defined to be

\[
Df(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}.
\]

Notes: this is not the Jacobian, which we’ll see later. The book doesn’t call it the Jacobian matrix, they just call it the matrix of the total

©2012, T. Vogel