Series with Non-negative Terms (6.2)

The question which will generally be asked is “Does a series converge?”, not “What does its sum equal?” This is for two reasons. First, if we know it converges, we can often get an approximation to the sum by looking at partial sums (although if it converges too slowly, this might not be practical), and second, it’s generally really hard or impossible to find the actual sum if it converges. In this section, we’ll be looking at series with positive terms, i.e., $a_k \geq 0$, and coming up with some convergence tests.

**Theorem:** If $a_k \geq 0$ for all $k$, then the following two are equivalent:

1. The sequence of partial sums $\{\sum_{k=1}^{n} a_k\}$ is bounded.
2. $\sum_{k=1}^{\infty} a_k$ converges.

**Proof:** Certainly $2 \Rightarrow 1$, since if the series converges, then the sequence of partial sums converges, and a convergent sequence is bounded. Now for $1 \Rightarrow 2$. Notice that if all of the $a_k$’s are positive, then the sequence of partial sums is increasing:

$$s_{n+1} - s_n = \sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n} a_k = a_{n+1} \geq 0.$$ 

But we know that a bounded, monotone sequence converges.

**Note:** $\sum_{k=1}^{\infty} (-1)^k$ has bounded partial sums, but does not converge. This doesn’t contradict the theorem, of course, since the terms aren’t positive.

First application of the theorem is the **Integral Test**.

**Theorem:** Suppose that $f(x)$ is positive and decreasing on $[1, \infty)$. Then $\sum_{k=1}^{\infty} f(k)$ converges iff $\int_{1}^{\infty} f(x) \, dx$ converges. (Note: the “1” isn’t that important: we could start the sum anywhere.)

**Proof:** First, $f(k) \geq \int_{k}^{k+1} f(x) \, dx$ (picture), since $f$ is decreasing, so

$$\sum_{k=1}^{n} f(k) \geq \sum_{k=1}^{n} \int_{k}^{k+1} f(x) \, dx = \int_{1}^{n+1} f(x) \, dx.$$ 

If $\int_{1}^{\infty} f(x) \, dx$ diverges, then $\lim_{N \to \infty} \int_{1}^{N} f(x) \, dx = \infty$ (using the fact that $f(x)$ is positive here), so the sequence of partial sums heads off
to infinity as well, and the series diverges. The contrapositive is also true: if the series converges, the integral converges. This is half the theorem. On the other hand, \( f(k) \leq \frac{1}{k} \int_{k-1}^{k} f(x) \, dx \) (picture). Thus, if the improper integral converges,

\[
\sum_{k=1}^{n} f(k) \leq f(1) + \int_{1}^{n} f(x) \, dx \leq f(1) + \int_{1}^{\infty} f(x) \, dx,
\]

which bounds the sequence of partial sums, implying that the series converges. Again, the contrapositive is also true: if the series diverges, so does the integral.

Important: this doesn’t imply that \( \int_{1}^{\infty} f(x) \, dx \) actually equals \( \sum_{k=1}^{\infty} f(k)! \). Just that these guys are either both finite or both infinite.

Main application of the integral test is \( p \)-series. Recall that \( \int_{1}^{\infty} \frac{1}{x^p} \, dx \) converges if \( p > 1 \). Then, at least for \( p > 0 \) we can apply the integral test to say that \( \sum_{k=1}^{\infty} \frac{1}{k^p} \) diverges for \( p \leq 1 \) and converges for \( p > 1 \), since \( \frac{1}{x^p} \) is positive and decreasing for those \( p \)'s. (For \( p \leq 0 \) we can’t apply the integral test, since the function is not decreasing. On the other hand, it obviously doesn’t go to zero for those \( p \), so the series diverges by the fact that the terms don’t go to zero.) The special case \( p = 1 \) is \( \sum_{k=1}^{\infty} \frac{1}{k} \) is known as the harmonic series, which is divergent.

Note: you really need that \( f \) is decreasing (at least for all \( x \) sufficiently large), not just positive. As an example, let \( f(x) = \sin^2 \pi x \). Then \( \sum_{k=1}^{\infty} f(k) = \sum_{k=1}^{\infty} 0 = 0 \), but it’s not hard to show that \( \int_{1}^{\infty} \sin^2 \pi x \, dx \) diverges (picture).

The main reason for the integral test is to get \( p \)-series. These are often used in comparisons, which we’ll do next.

**Theorem:** (Basic Comparison Test, or BCT) Suppose that \( 0 \leq a_k \leq b_k \) for \( k \) sufficiently large. Then:

1. if \( \sum_{k=1}^{\infty} b_k \) converges, then so does \( \sum_{k=1}^{\infty} a_k \).
2. if \( \sum_{k=1}^{\infty} a_k \) diverges, then so does \( \sum_{k=1}^{\infty} b_k \).

For simplicity, assume that \( 0 \leq a_k \leq b_k \) holds for all \( k \): neglecting a finite number of terms won’t affect convergence. For 1, we need to bound partial sums of \( \sum a_k \). We have

\[
\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k \leq \sum_{k=1}^{\infty} b_k
\]

which gives a bound for all of the partial sums of \( \sum a_k \), showing convergence. 2 is the contrapositive of 1, hence is logically equivalent. Or, in other words, suppose that \( \sum a_k \) diverges. Could \( \sum b_k \) converge? No, by part 1.

©2012, T. Vogel
Example 4. Does $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ converge?

For $k > e$, $\ln k > 1$, so $\frac{\ln k}{k} > \frac{1}{k}$. Since $\sum \frac{1}{k}$ diverges, so does the given series by the basic comparison test.

Example 5. Does $\sum_{k=1}^{\infty} \frac{\ln k}{k^{3/2}}$ converge?

The BCT with $\sum \frac{1}{k^{3/2}}$ does no good: being greater than a convergent series says nothing. Instead, pull off a small power to take care of $\ln k$:

$$\frac{\ln k}{k^{3/2}} = \frac{1}{k^{5/4}} \cdot \frac{\ln k}{k^{1/4}}.$$

Now,

$$\lim_{x \to \infty} \frac{\ln x}{x^{1/4}} = \lim_{x \to \infty} \frac{1}{4x^{-3/4}} = \lim_{x \to \infty} \frac{4}{x^{1/4}} = 0,$$

so for $k$ sufficiently large, $\frac{\ln k}{k^{3/2}} < 1$ (specifically, $k > 5504$.) Thus, for $k$ sufficiently large,

$$\frac{\ln k}{k^{3/2}} = \frac{1}{k^{5/4}} \cdot \frac{\ln k}{k^{1/4}} < \frac{1}{k^{5/4}}.$$

Since $\sum \frac{1}{k^{3/2}}$ is a convergent $p$-series, the original series converges by the basic comparison test.

Another comparison test is also often useful:

**Theorem**: (Limit Comparison Test, or LCT) Suppose that $a_k \geq 0$, $b_k > 0$ for all sufficiently large $k$, and that $L = \lim_{n \to \infty} \frac{a_n}{b_n}$ exists (as an extended real, i.e., could be $+\infty$. Then

1. If $0 < L < \infty$ then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
2. For $L = 0$, if $\sum b_k$ converges, then so does $\sum a_k$. If $\sum a_k$ diverges, then so does $\sum b_k$.
3. For $L = \infty$, if $\sum a_k$ converges, then so does $\sum b_k$. If $\sum b_k$ diverges, then so does $\sum a_k$.

Note: in case 2, it’s possible to have $\sum a_k$ converging and $\sum b_k$ diverging. In case 3, it’s possible to have $\sum b_k$ converging and $\sum a_k$ diverging.

**Proof**:

1. $(L \in (0, \infty))$ Take $\epsilon = \frac{L}{2}$. There exists $N$ so that $n > N$ implies that

$$\left| \frac{a_n}{b_n} - L \right| < \frac{L}{2},$$

i.e.,

$$0 < \frac{1}{2}L < \frac{a_n}{b_n} < \frac{3}{2}L,$$

©2012, T. Vogel
i.e.,

\[ 0 < \frac{1}{2} Lb_n < a_n < \frac{3}{2} Lb_n. \]

If \( \sum a_k \) converges, then \( \sum \frac{1}{2} Lb_k \) converges by BCT, hence \( \sum b_k \) converges (multiplying a convergent series by \( \frac{1}{2} \) gives a convergent series). If \( \sum a_k \) diverges, then BCT gives that \( \sum \frac{3}{2} Lb_k \) diverges, hence \( \sum b_k \) diverges.

(2) \( L = 0 \). For \( k \) sufficiently large, \( 0 < \frac{a_k}{b_k} < 1 \), so \( a_k < b_k \) for \( k \) sufficiently large. The result follows from BCT.

(3) \( L = \infty \). For \( k \) sufficiently large, \( \frac{a_k}{b_k} > 1 \), so \( a_k > b_k \) for \( k \) sufficiently large. The result follows from BCT.

**Example 6.** Does \( \sum_{k=1}^{\infty} \frac{k+1}{k^2+k+1} \) converge?

For large \( k \), the fraction “looks like” \( \frac{1}{k} \), so we’re inspired to use LCT with \( \sum \frac{1}{k} \):

\[ \lim_{k \to \infty} \frac{1/k}{(k+1)/(k^2+k+1)} = \lim_{k \to \infty} \frac{k^2+k+1}{k^2+k} = 1, \]

so the series act the same by LCT: they either both converge or both diverge. Since \( \sum \frac{1}{k} \) diverges, so does \( \sum_{k=1}^{\infty} \frac{k+1}{k^2+k+1} \).

**Example 7.** Does \( \sum_{k=1}^{\infty} \left( e^{1/k^2} - 1 \right) \) converge?

Intuitively, \( e^x \approx 1+x+\frac{1}{2}x^2+\cdots \), so maybe for small \( x \), \( e^x - 1 \) is “like” \( x \). This inspires us to use the LCT with \( \sum \frac{1}{k^2} \). I’m not eager to evaluate \( \lim_{k \to \infty} \frac{e^{1/k^2}-1}{1/k^2} \) directly. Instead, since \( \frac{1}{k^2} \to 0^+ \), if \( \lim_{x \to 0^+} \frac{e^x-1}{x} \) exists, it must equal \( \lim_{k \to \infty} \frac{e^{1/k^2}-1}{1/k^2} \). But \( \lim_{x \to 0^+} \frac{e^x-1}{x} \) is the definition of derivative of \( e^x \) at 0 (or use l’Hôpital’s rule) and must be 1. Therefore we get a 1 for LCT, and \( \sum_{k=1}^{\infty} \left( e^{1/k^2} - 1 \right) \) must act the same way as \( \sum \frac{1}{k^2} \), hence converges.