Alternating Series (6.4)

Will not follow book. Instead, I’ll give a more elementary proof of the Alternating series test which does not use Abel’s formula (but we will see Abel’s formula later).

**Theorem:** (Alternating Series Test) Suppose that \( \{a_k\} \) decreases monotonically to zero. Then \( \sum_{k=1}^{\infty} (-1)^k a_k \) converges.

**Proof:** Look at the sequence \( s_n = \sum_{k=1}^{n} (-1)^k a_k \) of partial sums. In particular, consider the subsequences of even indices and odd indices:

\[
s_1 = -a_1
\]
\[
s_3 = -a_1 + (a_2 - a_3)
\]
\[
s_5 = -a_1 + (a_2 - a_3) + (a_4 - a_5)
\]
\[\ldots\]
\[
s_{2k+1} = s_{2k-1} + (a_{2k} - a_{2k+1}) > s_{2k-1},
\]

so the subsequence of odd partial sums is increasing. Similarly,

\[
s_2 = -a_1 + a_2 = -(a_1 - a_2)
\]
\[
s_4 = -(a_1 - a_2) - (a_3 - a_4)
\]
\[\vdots\]
\[
s_{2k} = s_{2k-1} - (a_{2k-1} - a_{2k}) < s_{2k-2},
\]

and the subsequence of even partial sums is decreasing. Finally, I claim that every even partial sum is greater than every odd partial sum. Look at \( s_{2n}, s_{2m+1} \). Assume that \( 2m + 1 > 2n \) (the other case is similar). Then

\[
s_{2m+1} = s_{2n} - a_{2n+1} + a_{2n+2} - \cdots - a_{2m+1}
\]
\[
= s_{2n} - (a_{2n+1} - a_{2n+2}) - (a_{2n+3} - a_{2n+4}) - \cdots - (a_{2m-1} - a_{2m}) - a_{2m+1}.
\]

Since I’m subtracting off a bunch of positive terms, \( s_{2m+1} < s_{2n} \).

So, \( \{s_{2k+1}\} \) is increasing, bounded above by any even partial sum, say \( s_2 \), so it has a limit, call it \( L_1 \). Similarly, \( \{s_{2k}\} \) is decreasing, bounded below by any odd partial sum, hence has a limit, call it \( L_2 \).
All that’s left is to show that these are equal. But
\[
0 = \lim_{n \to \infty} a_{2n} = \lim_{n \to \infty} (s_{2n} - s_{2n+1}) = L_2 - L_1,
\]
so these limits are the same, and the sequence of partial sums converges.

Note: the even partial sums are always above the sum of the series (since they decrease down to the sum), and the odd partial sums are always below the sum of the series. Thus the sum of the series always lies between successive partial sums. (picture) As a consequence, if \( \sum_{k=1}^{\infty} (-1)^k a_k \) converges by satisfying the AST, then for any \( N \) we have
\[
\left| \sum_{k=1}^{\infty} (-1)^k a_k - \sum_{k=1}^{N} (-1)^k a_k \right| < a_{N+1}.
\]

**Example 12.** Show that \( \sum_{k=2}^{\infty} \frac{(-1)^k}{\log k} \) converges, and find \( N \) so that we’re guaranteed that
\[
\left| \sum_{k=2}^{\infty} \frac{(-1)^k}{\log k} - \sum_{k=2}^{N} \frac{(-1)^k}{\log k} \right| < 10^{-2}.
\]

The series obviously alternates, but we must check that \( \frac{1}{\log k} \) not only goes to zero as \( k \) goes to infinity, but actually decreases monotonically to zero. That the limit of \( \frac{1}{\log k} \) is zero as \( k \) goes to infinity is no problem, but what about the monotonicity? If \( f(x) = \frac{1}{\log x} = (\log x)^{-1} \), then
\[
f'(x) = - (\log x)^{-2} \frac{1}{x},
\]
which is negative for \( x \) greater than zero. Therefore \( f(x) \) decreases, and since the sequence \( \left\{ \frac{1}{\log k} \right\} \) is obtained by plugging integers into \( f(x) \), this sequence is monotonically decreasing. Thus the series converges by the alternating series test. Now, how many terms do we have to add up in a partial sum to be sure that we’re within \( 10^{-2} \) of the sum of the series? From equation (1), we will be sure if we have \( N \) so that \( a_{N+1} \) is less than \( 10^{-2} \). A little work gives us that for \( N \) to satisfy this we need \( N > e^{100} - 1 \), which is about \( 2.7 \times 10^{43} \).

I’ll postpone Abel’s formula and Dirichlet’s test until section 7.2, which will put them in a more reasonable context.