Anyway, back to the theorem:

**Proof:**

1. Suppose that the collection of open sets is $V_\alpha$, $\alpha \in A$. Take $x \in \bigcup_{\alpha \in A} V_\alpha$. Then $x$ is in at least one of the $V_\alpha$’s, call it $V_{\alpha_0}$. Since $V_{\alpha_0}$ is open and contains $x$, there is an $\varepsilon > 0$ so that $B_\varepsilon(x) \subseteq V_{\alpha_0}$. But $V_{\alpha_0} \subseteq \bigcup_{\alpha \in A} V_\alpha$, so $B_\varepsilon(x) \subseteq \bigcup_{\alpha \in A} V_\alpha$ as well. What we’ve shown is that every $x \in \bigcup_{\alpha \in A} V_\alpha$ is the center of an open ball contained in $\bigcup_{\alpha \in A} V_\alpha$, thus $\bigcup_{\alpha \in A} V_\alpha$ is open.

2. Suppose that $V_1, \ldots, V_k$ are open sets, and that $x \in \bigcap_{j=1}^k V_j$. Then $x$ is in each $V_j$. For each $V_j$, there is an $\varepsilon_j > 0$ so that $B_{\varepsilon_j}(x) \subseteq V_j$, since each $V_j$ is open. Let $\varepsilon = \min(\varepsilon_1, \ldots, \varepsilon_k)$. Since there are only finitely many $\varepsilon_j$’s, $\varepsilon$ must be one of them, hence $\varepsilon > 0$. But then $B_{\varepsilon}(x) \subseteq B_{\varepsilon_j}(x) \subseteq V_j$ for all $j$, thus $B_{\varepsilon}(x) \subseteq \bigcap_{j=1}^k V_j$, hence $\bigcap_{j=1}^k V_j$ is open.

3. Suppose that $E_1, E_2, \ldots, E_k$ are all closed sets. By definition, their complements $E_j^c$ are all open sets. From deMorgan’s laws (again see section 1.4),

$$\left( \bigcup_{j=1}^k E_j \right)^c = \bigcap_{j=1}^k (E_j^c).$$

By part 2, the right hand side is an open set, so $\left( \bigcup_{j=1}^k E_j \right)^c$ is open, hence $\bigcup_{j=1}^k E_j$ is closed.

4. Suppose that each $E_\alpha$, $\alpha \in A$ is closed. Then

$$\left( \bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} (E_\alpha^c).$$

By part 1, the right hand side is open, hence $\bigcap_{\alpha \in A} E_\alpha$ is closed.

**Note:** Arbitrary intersections of open sets need not be open. Arbitrary unions of closed sets need not be closed. We saw this in the example that I did earlier. Here are a couple more examples:

©2012, T. Vogel
• Let $V_k = \left( -\frac{1}{k}, \frac{1}{k} \right)$, for $k \in \mathbb{N}$, so these are all open. However, $\bigcap_{k=1}^{\infty} V_k = \{0\}$, which is not an open set.

• Let $E_k = \left[ \frac{1}{k}, \infty \right)$ for all $k \in \mathbb{N}$, so that these are all closed sets. However, $\bigcup_{k=1}^{\infty} E_k = (0, \infty)$ which is not a closed set.

**Note:** Here's one type of set in $\mathbb{R}^2$ which we can say is open: if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $E = \{(x, y) \mid y > f(x)\}$ is open. Proof: if not, then there is a point $(x_0, y_0) \in E$ so that there is no $\varepsilon$ ball around $(x_0, y_0)$ entirely contained in $E$. If you set $\varepsilon$ to be $\frac{1}{k}$, you find that there is a sequence of points $(x_k, y_k)$ with $x_k \to x_0$, $y_k \to y_0$ for which $(x_k, y_k) \notin E$. By definition of $E$, this means that we have $y_k \leq f(x_k)$. But taking $k$ to infinity in this relationship and using continuity implies $y_0 \leq f(x_0)$, a contradiction.

One reason that open sets are important in analysis:

**Proposition:** $f : \mathbb{R} \to \mathbb{R}$ is continuous iff inverse images of open sets are open. (Recall inverse images, section 1.5: $x \in f^{-1}(A)$ iff $f(x) \in A$. For example, if $f(x) = x^2$, then $f^{-1}([-1, 4]) = [-2, 2]$ (picture))

**Proof:**

"$\Rightarrow$": Suppose $f$ is continuous, and let $\Omega$ be open. By definition, $f^{-1}(\Omega)$ is the set of all $x \in \mathbb{R}$ so that $f(x) \in \Omega$. Take such an $x_0$. Since $f(x_0) \in \Omega$ and $\Omega$ is open, there is an $\varepsilon > 0$ so that $B_\varepsilon(f(x_0)) \subseteq \Omega$. Since $f$ is continuous at $x_0$, there is a $\delta > 0$ so that $|x_0 - y| < \delta$ implies that $|f(x_0) - f(y)| < \varepsilon$, i.e., $f(y) \in B_\varepsilon(f(x_0))$. This implies that $y \in f^{-1}(\Omega)$. Since this was true for every $y$ within $\delta$ of $x_0$, we have $B_\delta(x_0) \subseteq f^{-1}(\Omega)$. Therefore $f^{-1}(\Omega)$ is open.

"$\Leftarrow$": Suppose that inverse images of open sets are open. Take $x_0 \in \mathbb{R}$. We want to show continuity at $x_0$, by the $\delta, \varepsilon$ definition. Pick an $\varepsilon > 0$, and consider the set $B_\varepsilon(f(x_0))$. This is an open set, hence by hypothesis $f^{-1}(B_\varepsilon(f(x_0)))$ is open. One element of this set is $x_0$ itself. Therefore there is a $\delta > 0$ so that $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$. Now unravel this statement: everything within $\delta$ of $x_0$ gets mapped by $f$ to the $\varepsilon$ ball around $f(x_0)$. That’s saying precisely that $|x_0 - y| < \delta$ implies $|f(x_0) - f(y)| < \varepsilon$. Thus $f$ is continuous at $x_0$, and since $x_0$ was arbitrary, $f$ is continuous on $\mathbb{R}$.

We’ll see that this generalizes to continuous functions on $\mathbb{R}^n$, but we haven’t defined those yet.

It was important that the domain of $f$ in the proposition was $\mathbb{R}$. For example, if $f : [0, 1] \to \mathbb{R}$ is given by $f(x) = 2x$, then $f^{-1}((1, 4)) = \left( \frac{1}{2}, 1 \right]$. We can still get a true proposition with a slight modification: we’ll see the proof in chapter 9.