cover $H$. This will be based on the set $T$ of open balls with rational radii and centers in $Q^n$. Then we’ll show that this countable subcover actually admits a finite subcover. This will be based on the Bolzano-Weierstrass Theorem.

**Proof of Borel Covering Lemma:** For each $x \in H$, consider the ball $B_r(x)$ $(x)$. The collection of all such balls pretty clearly covers $H$, since each point of $H$ is the center of one of these balls. Take the ball of $T$ promised by 2, above, for $B_r(x)$ $(x)$. This ball covers $x$, and if we do this for each $x \in H$, we cover $H$ by balls in $T$. But $T$ is countable, so at this point we can say that there is a countable collection of balls which cover $H$: call them $B_q_1(a_1), B_q_2(a_2), \ldots$, (for simplicity $B_1,B_2,\cdots$) corresponding to $x_1,x_2,\cdots$ in $H$. (Note that this gives a countable collection of the original balls which cover $H$, since $B_i \subseteq B_r(x_i)(x_i)$, but we’re going to stick with the $B_i$’s). I claim that there is a finite subcover of $H$ of these balls $B_i$ of $T$. Suppose not. Then in particular, for any $k$, the collection $B_1,\cdots, B_k$ does not cover $H$, so there is a point $x_k \in H - \bigcup_{i=1}^k B_k$. Since $H$ is bounded, the sequence $\{x_k\}$ is bounded, and therefore by Bolzano-Weierstrass, there is a convergent subsequence $\{x_{k_j}\}$ which converges to some point $a$. Since $H$ is closed, the point $a$ must be in $H$. But $H$ is covered by the collection $\{B_j\}$, so in particular $a \in B_m$ for some $m$. Since $B_m$ is open and $x_{k_j} \to a$, we must have $x_{k_j} \in B_m$ for $j$ sufficiently large. But this can’t happen: once $k_j$ is larger than $m$, $x_{k_j} \in H - \bigcup_{i=1}^{k_j} B_i$, and we’ve subtracted off $B_m$ in this. Therefore there is a finite subcollection of the balls $B_{q_1}(a_1), B_{q_2}(a_2), \cdots$ which cover $H$. Finally, each of these balls satisfy $B_{q_i}(a_i) \subseteq B_r(x_i)(x_i)$, so we get the finite collection of balls promised.

**Theorem (Hard part of Heine-Borel Theorem):** If $H \subseteq \mathbb{R}^n$ is closed and bounded, $H$ is compact.

**Proof:** Suppose that we have an open cover $U_\alpha, \alpha \in A$ of $H$. Every point $x \in H$ is covered by at least one of these sets. So, for each $x$, put it inside one of the $U_\alpha$’s, call it $U_{\alpha}(x)$. Since $U_{\alpha}(x)$ is open, there is an $r(x) > 0$ so that $B_r(x)(x) \subseteq U_{\alpha}(x)$. This is the function $r$ that we want to apply the Borel Covering lemma to. By the lemma, there is a finite collection $B_{r(x_i)}(x_i), i = 1, \cdots, m$ which cover $H$. But $B_{r(x_i)}(x_i) \subseteq V_{\alpha(x_i)}$, so $\{V_{\alpha(x_i)}\}, i = 1, \cdots, m$ covers $H$. Thus every open cover of $H$ admits a finite subcover, and $H$ is compact.

**Note:**

We need both closed and bounded to conclude that a subset of $\mathbb{R}^n$ is compact. Simply bounded is not enough: we’ve seen that $(0,1)$ is not compact. Closed but not bounded is not enough either: $[0,\infty)$ is covered by $\{(-1,n)\}, n \in \mathbb{N}$, but there is no finite subcover.

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