Limits of functions (9.3)

First off, we’re thinking of *vector-valued functions*, i.e., \( f : U \to \mathbb{R}^m \), \( U \subseteq \mathbb{R}^n \).

**Example:**
Let \( f(x, y) = \frac{xy}{x^2 + y^2} \). This is defined for all \((x, y) \neq (0, 0)\), so domain \( U \) of \( f \) is \( U = \mathbb{R}^2 - (0, 0) \). (Note: I should probably be writing \( f(x, y) \) instead, since I want to think of vectors in \( \mathbb{R}^n \) as column vectors. Well, sometimes I will and sometimes I won’t.) This \( f \) maps \( U \) into \( \mathbb{R}^1 \).

What would a function \( g \) which maps a subset of \( \mathbb{R}^2 \) into \( \mathbb{R}^2 \) look like? (Notation note: In these notes, I’ll use \( g \) for a vector-valued function to agree with the book’s notation. On the other hand, I don’t know how to do bold-faced on a chalk-board, so I’ll write it as \( \vec{g} \) in class.) Well, since we’re getting vectors out, it would be something like

\[
\vec{g} \left( \begin{array}{c}
x \\
y \
\end{array} \right) = \left( \begin{array}{c}
xy \\
x^2 + 3xy \
\end{array} \right).
\]

In general, if \( g : U \subseteq \mathbb{R}^n \to \mathbb{R}^m \), \( g \) will have \( m \) component functions. We can write this as

\[
\vec{g} \left( \begin{array}{c}
x_1 \\
\vdots \\
x_n \
\end{array} \right) = \left( \begin{array}{c}
g_1(x) \\
\vdots \\
g_m(x) \
\end{array} \right).
\]

The domain of a vector valued function is generally taken to be the largest set of \( x \)'s for which all of the component functions are defined (unless this is specifically restricted).

**Example:** \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) given by

\[
f \left( \begin{array}{c}
x \\
y \
\end{array} \right) = \left( \begin{array}{c}
1/(1 - x^2 - y^2) \\
\ln(xy) \
\end{array} \right).
\]

For the first component, we need \( x^2 + y^2 \neq 1 \), i.e., everything except the unit circle. For the second component, need \( xy > 0 \), i.e., either both \( x \) and \( y \) are positive or both \( x \) and \( y \) are negative. So, the natural domain of \( f \) is everything inside the first and third quadrants, except for the quarters of the unit circles there. (sketch).
The definition of limit of a function is pretty much a generalization of what we saw in 409.

**Definition:** Suppose \( f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m \), \( L \in \mathbb{R}^m \), \( a \in \mathbb{R}^n \). Suppose that \( f \) is defined in some open ball \( V \) containing \( a \), except possibly at \( a \) itself. We say \( \lim_{x \to a} f(x) = L \) iff \( \forall \varepsilon > 0 \ \exists \delta > 0 \) so that \( 0 < \|x - a\| < \delta \) implies \( \|f(x) - L\| < \varepsilon \). (Picture for \( m = n = 2 \).)

**Example:** Suppose \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is given by \( f(x, y) = (2x, 3y) \).

Prove, straight from the definition, that \( \lim_{(x, y) \to (1, 1)} f(x, y) = (2, 3) \),

Note: for readability, I’ll write \( \lim_{(x, y) \to (1, 1)} f(x, y) = (2, 3) \), but we really think of vectors in \( \mathbb{R}^n \) as column vectors. I may also write it as \( \lim_{(x, y)^T \to (1, 1)^T} f(x, y)^T = (2, 3)^T \),

where \( T \) means matrix transpose. Anyway.

Given \( \varepsilon > 0 \), we seek \( \delta > 0 \) so that \( 0 < \|(x, y) - (1, 1)\| < \delta \) implies \( \|(2x, 3y) - (2, 3)\| < \varepsilon \). I’ll use the idea of finding something that is i) larger than \( \|(2x, 3y) - (2, 3)\| \), ii) simpler than \( \|(2x, 3y) - (2, 3)\| \), and iii) still going to zero.

Since \( (2x - 2)^2 \leq (3x - 3)^2 \) for all \( x \), we can certainly say \( \|(2x, 3y) - (2, 3)\| = \sqrt{(2x - 2)^2 + (3y - 3)^2} \leq \sqrt{(3x - 3)^2 + (3y - 3)^2} = 3 \|(x, y) - (1, 1)\| \).

Since we want \( \|(2x, 3y) - (2, 3)\| \) to be less than \( \varepsilon \), this inspires us to take \( \delta \) to be \( \frac{\varepsilon}{3} \). Once we’ve done the above work, the proof is easy: given \( \varepsilon > 0 \), take \( (x, y) \) with \( 0 < \|(x, y) - (1, 1)\| < \frac{\varepsilon}{3} \). Then certainly \( \|(x, y) - (1, 1)\| < \frac{\varepsilon}{3} \), which implies \( \sqrt{(3x - 3)^2 + (3y - 3)^2} < \varepsilon \). But since \( \|(2x, 3y) - (2, 3)\| \leq \sqrt{(3x - 3)^2 + (3y - 3)^2} \), this implies \( \|(2x, 3y) - (2, 3)\| < \varepsilon \), as desired.

**Sequential characterization of limit of a function:**

\[
\lim_{x \to a} f(x) = L \quad \text{iff} \quad \lim_{k \to \infty} f(x_k) = L
\]

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66