Recall that, back in section 8.3, I proved that \( f : \mathbb{R} \to \mathbb{R} \) is continuous iff \( f^{-1}(\Omega) \) for every open set \( \Omega \). It’s time to generalize. First we need a lemma, though. This says that there is a statement about relatively open sets which is analogous to the definition of a (plain vanilla) open set. This is part of Remark 8.27, section 8.3, which we didn’t need until now.

**Lemma:** \( U \subseteq E \) is relatively open in \( E \) iff \( \forall a \in U \) there exists \( \varepsilon > 0 \) so that \( B_\varepsilon(a) \cap E \subseteq U \).

**Proof:** "\( \Rightarrow \)" First suppose that \( U \) is relatively open in \( E \). Then there exists an open set \( A \) so that \( U = E \cap A \). Since \( A \) is open and \( U \) is a subset of \( A \), for any \( a \in U \) there is an \( \varepsilon > 0 \) so that \( B_\varepsilon(a) \subseteq A \). But then \( B_\varepsilon(a) \cap E \subseteq A \cap E = U \).

"\( \Leftarrow \)" Conversely, suppose that \( U \) has the property that \( \forall a \in U \) there exists \( \varepsilon > 0 \) so that \( B_\varepsilon(a) \cap E \subseteq U \). For each \( a \in U \), find the appropriate \( \varepsilon \), and call it \( \varepsilon(a) \). Set \( A = \bigcup_{a \in A} B_\varepsilon(a) \). This is certainly an open set, since it’s the union of open sets. I claim that \( U = A \cap E \). Certainly if \( a \in U \), then \( a \in B_\varepsilon(a) \), hence in \( A \). Also, since \( U \subseteq E \), \( a \in E \). Thus \( a \in A \cap E \), and \( U \subseteq A \cap E \). For the other containment, take \( x \in A \cap E \). Since \( x \in A \), \( x \) must be in \( B_\varepsilon(a) \) for some \( a \in A \). Since \( x \in E \) as well, \( x \in B_\varepsilon(a) \cap E \). But by choice of \( \varepsilon \), we must have \( x \in U \).

**Theorem:** \( f : E \to \mathbb{R}^m \) is continuous on \( E \) iff \( f^{-1}(\Omega) \) is relatively open in \( E \) for every open set \( \Omega \) in \( \mathbb{R}^m \).

**Proof:** First suppose that \( f \) is continuous on \( E \), and that \( \Omega \subseteq \) is open. If \( f^{-1}(\Omega) \) is empty, it’s open, so assume that \( f^{-1}(\Omega) \) is non-empty, and take \( a \in f^{-1}(\Omega) \). Then \( f(a) \in \Omega \). Since \( \Omega \) is open, there is \( \varepsilon > 0 \) so that \( B_\varepsilon(f(a)) \subseteq \Omega \). Since \( f \) is continuous at \( a \), there is \( \delta > 0 \) so that \( \|x - a\| < \delta \Rightarrow x \in E \) implies \( \|f(x) - f(a)\| < \varepsilon \). In other words, everything in \( B_\delta(a) \cap E \) gets taken by \( f \) to within \( \varepsilon \) of \( f(a) \), hence into \( \Omega \). This says that \( B_\delta(a) \cap E \subseteq f^{-1}(\Omega) \). This is true for arbitrary \( a \in f^{-1}(\Omega) \), so by the lemma, \( f^{-1}(\Omega) \) is relatively open in \( E \).

Now suppose that inverse images of open sets are relatively open. Take an arbitrary \( a \in E \) and an \( \varepsilon > 0 \). We have that \( B_\varepsilon(f(a)) \) is an open set, hence \( f^{-1}(B_\varepsilon(f(a))) \) is relatively open in \( E \). Since \( a \in f^{-1}(B_\varepsilon(f(a))) \), by the lemma, there is a \( \delta > 0 \) so that \( B_\delta(a) \cap E \subseteq f^{-1}(B_\varepsilon(f(a))) \). Now untangle this statement. What it’s saying is that if you take \( f \) of any point of \( E \) that’s within \( \delta \) of \( a \), you must end up in the \( \varepsilon \) ball around \( f(a) \). This is just the statement that \( x \in E \), \( \|x - a\| < \delta \) implies \( \|f(x) - f(a)\| < \varepsilon \), i.e., that \( f \) is continuous at \( a \), and hence on \( E \) since \( a \) was arbitrary.
Proposition: \( f^{-1}(\Omega) \) is relatively open for each open set \( \Omega \subseteq \mathbb{R}^m \) iff \( f^{-1}(C) \) is relatively closed for each closed set \( C \subseteq \mathbb{R}^m \). This gives a third way of characterizing continuity, but you don’t see it nearly as much as the statement about relatively open sets.

Proof: "\( \implies \)" Take a closed set \( C \). Then \( C^c \) is open, thus \( f^{-1}(C^c) \) is relatively open in \( E \). But I claim that \( f^{-1}(C^c) = E - f^{-1}(C) \). It’s just chasing the definition: \( a \in f^{-1}(C^c) \) iff \( f(a) \in C^c \), i.e., \( f(a) \notin C \), which is the same as \( a \notin f^{-1}(C) \), i.e., \( a \in E - f^{-1}(C) \). So, \( E - f^{-1}(C) \) is relatively open in \( E \), which we’ve seen is the same as \( f^{-1}(C) \) is relatively closed in \( E \).

"\( \impliedby \)" Take an open set \( \Omega \) and look at complements, just as in the previous part.

Note: Images of open sets under continuous maps need not be open. Here’s an example.

Example: \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = \sin x \). Then \( f((0,2\pi)) = [-1,1] \).

We can say something about images of compact and connected sets, however.

Theorem: If \( H \) is compact, \( f : H \to \mathbb{R}^m \) continuous, then \( f(H) \) is compact.

Proof: Suppose that \( \{V_\alpha : \alpha \in A\} \) is an open cover of \( f(H) \). We want to show that there is a finite subcover. I claim that the collection \( \{f^{-1}(V_\alpha) : \alpha \in A\} \) covers \( H \). The reason: if \( x \in H \), then \( f(x) \in f(H) \), so \( f(x) \in V_{\alpha_0} \) for some \( \alpha_0 \), thus \( x \in f^{-1}(V_{\alpha_0}) \). Now, the collection \( \{f^{-1}(V_\alpha) : \alpha \in A\} \) is not an open cover of \( H \), since all we know about each \( f^{-1}(V_\alpha) \) is that they are relatively open in \( H \), but this will be enough. Since each \( f^{-1}(V_\alpha) \) is relatively open in \( H \), there is an open set \( U_\alpha \) so that \( U_\alpha \cap H = f^{-1}(V_\alpha) \). The collection \( \{U_\alpha : \alpha \in A\} \) is an open cover of \( H \). Therefore, there is a finite subcover \( U_{\alpha_1}, \ldots, U_{\alpha_k} \). Since these cover \( H \), clearly their intersections with \( H \) cover \( H \), so \( f^{-1}(V_{\alpha_1}), \ldots, f^{-1}(V_{\alpha_k}) \) cover \( H \). But then \( V_{\alpha_1}, \ldots, V_{\alpha_k} \) cover \( f(H) \): \( y \in f(H) \) iff \( \exists x \in H \) so that \( f(x) = y \) (maybe more than \( 1 \) \( x \)), so there is \( V_{\alpha_j} \) with \( x \in f^{-1}(V_{\alpha_j}) \), so \( y \in V_{\alpha_j} \).

Note:

- Continuous images of closed sets need not be closed: let \( f(x) = \frac{1}{x} \). If \( C = [1,\infty) \) (closed, not compact), then \( f(C) = (0,1] \) is not closed.
- Continuous images of bounded sets need not be bounded: take the same \( f \) as before, and look at \( f((0,1]) \). This is \( [1,\infty) \).

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